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Trigonometry of spacetimes: a new self-dual approach to a curvature/signature (in)dependent trigonometry

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Abstract. A new method to obtain trigonometry for the real spaces of constant curvature and metric of any (even degenerate) signature is presented. The method could be described as 'curvature/signature (in)dependent trigonometry' and encapsulates trigonometry for all these spaces into a single *basic trigonometric group equation*. This brings to its logical end the idea of an 'absolute trigonometry', and provides equations which hold true for the nine two-dimensional spaces of constant curvature and any signature. This family of spaces includes both relativistic and non-relativistic spacetimes; therefore a complete discussion of trigonometry in the six de Sitter, Minkowskian, Newton–Hooke and Galilean spacetimes follow as particular instances of the general approach. Distinctive traits of the method are 'universality' and 'self-duality': every equation is meaningful for the nine spaces at once, and displays invariance explicitly under a duality transformation relating the nine spaces amongst themselves. These basic structural properties allow a complete study of trigonometry and, in fact, *any* equation previously known for the three classical (Riemannian) spaces also has a version for the remaining six 'spacetimes'; in most cases these equations are new.

1. Introduction

The trigonometry of relativistic homogeneous, constant-curvature models of spacetimes (anti-de Sitter, Minkowski and de Sitter) is the most elementary part of the geometry in these spacetimes. However, it has yet to become part of common knowledge in mathematics or theoretical physics. Trigonometry in Minkowskian spacetime was first studied explicitly by Birman and Nomizu [1] and except for some results in Yaglom's book [2], in which they are termed cohyperbolic and doubly hyperbolic geometries, we have not found any explicit formulation for the trigonometry in either anti-de Sitter or de Sitter spacetimes, in spite of the very basic nature and strong current interest in these spaces. Thus a first and short-term aim of this paper is to fill this gap.

There is also a second, more long-term aim. Trigonometry, the study of the simplest geometrical configuration in a given space, should be a basic building block within the specific study of the geometry of homogeneous symmetric spaces. Hence this paper should also (and in the long term, mainly) be read as a step within the general programme of studying the trigonometry of symmetric spaces (see [3–6]).

Within this perspective, the final and primary aim of this work is to introduce the ideas and methods of a group-theoretical derivation to trigonometry which we believe to be new. This approach does not consider trigonometry for a *single* space (for, say, the anti-de Sitter spacetime), but instead is addressed towards providing simultaneously the trigonometry of a *whole family* of spaces. This approach carries to its logical end the 'absolute trigonometry', first discussed by Bolyai and then continued by de Tilly amongst others [7, 8], which covered simultaneously the three classical spaces of constant curvature (sphere, Euclidean space and Lobachewski hyperbolic space).

There are several distinctive traits in this approach. First, economy of thought: a single (parameter-dependent) computation covers at once the trigonometry of several spaces. Second, a clear view is obtained of relationships between different spaces in the same family, such as several *dualities*; otherwise some of these may easily pass unnoticed, yet they may provide new insights. Third, limiting (contracted) cases, corresponding to vanishing curvature and/or a degenerate metric, are included and described at the same level as the generic ones, thus making completely redundant a separate study of *contractions*. These traits apply not only to trigonometry, but also to the study of most properties of geometries, groups and algebras within each family [9–18].

All symmetric homogeneous spaces can be classed into several natural families [13, 19, 20], each with their Lie groups of motion, Lie algebras, etc, which depend on some parameters distinguishing family members. In the (irreducible) spaces of real type, these parameters determine the *curvatures* and/or the *signatures* of the fundamental metric for each space in the family. Additional parameters in other families label a division algebra ($\mathbb{C}, \mathbb{H}, \mathbb{O}$) or a pseudo-division variant coordinatizing the space. The method we are proposing should furnish trigonometry for all these families of spaces.

In this paper we restrict ourselves to a complete and detailed discussion of the trigonometry of the rank-one symmetric homogeneous spaces of *real* type, called quadratic or orthogonal Cayley–Klein spaces (see, e.g., [12, 15, 21, 22]), that are associated with the quasi-orthogonal Lie groups $SO(N)$ and $SO(p, q)$ and some of their contractions; this will also serve as a background to underlie a forthcoming follow-up paper [23] which is devoted to the trigonometry of rank-one complex Hermitian spaces associated with the *unitary* groups.

Any three points in any rank-one real-type homogeneous symmetric space are always contained in a two-dimensional (2D) totally geodesic submanifold, so considering only 2D spaces (planes) is no restriction at all. There are *nine* 2D real quadratic Cayley–Klein spaces [2]: the sphere, Euclidean and hyperbolic planes, the co-Euclidean, Galilean and co-Minkowskian planes and finally the co-hyperbolic, Minkowskian and doubly hyperbolic planes. Only the first three spaces mentioned belong to the restricted family of the so-called *two-point homogeneous spaces* whose trigonometry is very well known. The remaining six spaces are not two-point homogeneous [24], but together with the three previous ones they provide a natural frame for a joint study of trigonometry. Within a concrete physical interpretation these six spaces are the (1 + 1)D symmetric homogeneous spacetimes: oscillating (or anti)Newton–Hooke, Galilean, expanding Newton–Hooke (1 + 1)D spacetimes, and anti-de Sitter, Minkowskian and de Sitter (1 + 1)D spacetimes. The trigonometry of the two constant-curvature counterparts of the special relativity spacetime, mentioned as a short-term first aim of this work, follows as a side effect. The required information on these nine 2D spaces is given in section 2.

The method we propose is presented in section 3. It embodies the trigonometry for the whole biparametric family of these real 2D Cayley–Klein spaces into a *single* group equation, which we call the *basic trigonometric identity*. Starting from this, our procedure allows a very rapid browsing through the complete *zoo* of trigonometric equations for the nine spaces which are obtained in section 4. These equations are very well known in the three constant-curvature *Riemannian* cases but we have not found any reference to the anti-de Sitter and de Sitter versions of most of these formulae, especially those involving areas and co-areas of triangular loops. In section 5 we translate some of the results to the kinematic language, and offer several trigonometric relations for the de Sitter and anti-de Sitter spacetimes, as well as for their non-relativistic analogues. A section with some final comments and prospects for continuation of this work closes the paper.

2. The nine two-dimensional real Cayley–Klein geometries

The motion groups of the nine 2D Cayley–Klein (CK) geometries of real type can be described in a unified setting by means of two real coefficients κ_1, κ_2 and are collectively denoted as $SO_{\kappa_1,\kappa_2}(3)$. The generators $\{P_1, P_2, J_{12}\}$ of the corresponding Lie algebras $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ have Lie commutators

$$[J_{12}, P_1] = P_2 \qquad [J_{12}, P_2] = -\kappa_2 P_1 \qquad [P_1, P_2] = \kappa_1 J_{12}. \tag{2.1}$$

There is a single Lie algebra Casimir coming from the Killing–Cartan form:

$$\mathcal{C} = P_2^2 + \kappa_2 P_1^2 + \kappa_1 J_{12}^2. \tag{2.2}$$

The CK algebras $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ can be endowed with a $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ group of commuting automorphisms generated by

$$\begin{aligned} \Pi_{(1)} &: (P_1, P_2, J_{12}) \to (-P_1, -P_2, J_{12}) \\ \Pi_{(2)} &: (P_1, P_2, J_{12}) \to (P_1, -P_2, -J_{12}). \end{aligned} \tag{2.3}$$

The two remaining involutions are the composition $\Pi_{(02)} = \Pi_{(1)} \cdot \Pi_{(2)}$ and the identity. Each involution Π determines a subalgebra of $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ the elements of which are invariant under Π; the subgroups generated by these subalgebras will be denoted by H.

The elements defining a 2D CK geometry are as follows [9, 10].

- The *plane* as the set of points corresponds to the 2D symmetrical homogeneous space

  $$S^2_{[\kappa_1],\kappa_2} \equiv SO_{\kappa_1,\kappa_2}(3)/H_{(1)} \equiv SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_2}(2) \qquad H_{(1)} = \langle J_{12}\rangle \approx SO_{\kappa_2}(2). \tag{2.4}$$

  The generator J_{12} leaves a point O (the origin) invariant, thus J_{12} acts as the rotation around O. The involution $\Pi_{(1)}$ is the reflection around the origin. In this space P_1 and P_2 generate translations which move the origin point in two basic directions.
- The set of *lines* is identified as the 2D symmetrical homogeneous space

  $$S^2_{\kappa_1,[\kappa_2]} \equiv SO_{\kappa_1,\kappa_2}(3)/H_{(2)} \equiv SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_1}(2) \qquad H_{(2)} = \langle P_1\rangle \approx SO_{\kappa_1}(2). \tag{2.5}$$

  In this space, the generator P_1 leaves invariant the 'origin' line l_1, which is moved in two basic directions by J_{12} and P_2. Therefore, within $S^2_{\kappa_1,[\kappa_2]}$, P_1 should be interpreted as the generator of 'rotations' around l_1, and the involution $\Pi_{(2)}$ is the reflection in l_1.
- There is a second set of lines corresponding to the 2D symmetrical homogeneous space

  $$SO_{\kappa_1,\kappa_2}(3)/H_{(02)} \equiv SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_1\kappa_2}(2) \qquad H_{(02)} = \langle P_2\rangle \approx SO_{\kappa_1\kappa_2}(2). \tag{2.6}$$

  In this case, P_2 leaves invariant an 'origin' line l_2 in this space, while J_{12} and P_1 do move l_2. The involution $\Pi_{(02)}$ is the reflection in the line l_2.

In order to distinguish the two sets of lines we will call the elements of $S^2_{\kappa_1,[\kappa_2]}$ *lines of the first kind*, while the elements of the space $SO_{\kappa_1,\kappa_2}(3)/H_{(02)}$ will be called *lines of the second kind*. By a *two-dimensional CK geometry* we will understand the set of three symmetrical homogeneous spaces of points, lines of the first kind and lines of the second kind. The group $SO_{\kappa_1,\kappa_2}(3)$ acts transitively on each of these spaces.

All properties of the two spaces of lines can be transcribed in terms of the space $S^2_{[\kappa_1],\kappa_2}$ itself, and in this interpretation the lines of the first or second kind can be seen as certain 1D submanifolds of $S^2_{[\kappa_1],\kappa_2}$ rather than 'points' in the spaces $S^2_{\kappa_1,[\kappa_2]}$ or $SO_{\kappa_1,\kappa_2}(3)/H_{(02)}$. In the following we will interpret everything in terms of the space $S^2_{[\kappa_1],\kappa_2}$, where l_1 and l_2 should be considered as two 'orthogonal' lines meeting in O.

The coefficients κ_1, κ_2 play a twofold role. The space $S^2_{[\kappa_1],\kappa_2}$ has a *quadratic metric* coming from the Casimir (2.2), the signature of which corresponds to the matrix $\mathrm{diag}(1, \kappa_2)$. This metric is Riemannian (definite positive) for $\kappa_2 > 0$, Lorentzian (indefinite) for $\kappa_2 < 0$ and degenerate for $\kappa_2 = 0$. This space has a canonical connection which is compatible with the metric, and has *constant curvature* equal to κ_1. In the notation $S^2_{[\kappa_1],\kappa_2}$, $S^2_{\kappa_1,[\kappa_2]}$ for the spaces, the κ_i in square brackets is the constant curvature, and the remaining constant determines the signature. Alternatively, the coefficients κ_1, κ_2 determine the *kind of measures of separation* amongst points and lines in the Klein sense [2, 9].

- The pencil of points on a first-kind line is elliptical/parabolic/hyperbolic according to whether κ_1 is greater than/equal to/less than zero.
- Likewise for the pencil of points on a second-kind line depending on the product $\kappa_1\kappa_2$.
- Likewise for the pencil of lines through a point according to κ_2.

For κ_1 positive/zero/negative the isotropy subgroup $H_{(2)}$ is $SO(2)/\mathbb{R}/SO(1,1)$, and the same happens for $H_{(1)}$ (respectively, $H_{(02)}$) according to the value of κ_2 (respectively, $\kappa_1\kappa_2 \equiv \kappa_{02}$). Whenever the coefficient κ_1 (respectively, κ_2) is different from zero, a suitable choice of length unit (respectively, angle unit) allows us to reduce it to either +1 or -1. Hence we obtain nine 2D real CK geometries which are displayed in table 1.

A fundamental property of the scheme of CK geometries is the existence of an 'automorphism' of the whole family, called *ordinary duality* $\mathcal{D}$, which is given by

$$\mathcal{D} : (P_1, P_2, J_{12}) \to (-J_{12}, -P_2, -P_1) \qquad \mathcal{D} : (\kappa_1, \kappa_2) \to (\kappa_2, \kappa_1). \tag{2.7}$$

The map $\mathcal{D}$ leaves the general commutation rules (2.1) invariant, while it interchanges the space of points with the space of first-kind lines, $S^2_{[\kappa_1],\kappa_2} \leftrightarrow S^2_{\kappa_1,[\kappa_2]}$, and the corresponding curvatures $\kappa_1 \leftrightarrow \kappa_2$, preserving the space of second-kind lines. Duality interchanges the Euclidean, hyperbolic and Minkowskian geometries with the co-Euclidean, co-hyperbolic and co-Minkowskian ones, while elliptic, Galilean and doubly hyperbolic are self-dual geometries. This also suggests a kind of duality between *curvature* and *signature* which would be worth studying.

The non-generic situation where a coefficient κ_i vanishes corresponds to an Inönü–Wigner contraction [25]. The limit $\kappa_1 \to 0$ is a local contraction (around a point), while the limit $\kappa_2 \to 0$ is an axial contraction (around a line) (see table 1). We remark that our approach to contractions is built-in to *any* expression associated with the CK geometries so that a contraction is simply equivalent to setting $\kappa_i = 0$ in the desired relation.

2.1. Spacetimes as Cayley–Klein spaces

Let $\mathcal{H}$, $\mathcal{P}$ and $\mathcal{K}$ be the generators of time translations, space translations and boosts, respectively, in the most simple (1 + 1)D homogeneous spacetime. Under the identification

$$P_1 \equiv \mathcal{H} \qquad P_2 \equiv \mathcal{P} \qquad J_{12} \equiv \mathcal{K} \tag{2.8}$$

the six CK groups with $\kappa_2 \leqslant 0$ (the second and third rows of table 1; NH denotes Newton–Hooke) are the motion groups of (1 + 1)D spacetimes [26]. The physical reading of the three CK spaces within each of these six CK geometries is as follows:

Table 1. The nine two-dimensional CK geometries.

	Measure of distance		
Measure of angle	Elliptic $\kappa_1 = 1$	Parabolic $\kappa_1 = 0$	Hyperbolic $\kappa_1 = -1$
Elliptic $\kappa_2 = 1$	Elliptic $SO(3)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = -P_1$ $[P_1, P_2] = J_{12}$ $\mathcal{C} = P_2^2 + P_1^2 + J_{12}^2$ $H_{(1)} = SO(2)$ $H_{(2)} = SO(2)$ $H_{(02)} = SO(2)$	Euclidean $ISO(2)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = -P_1$ $[P_1, P_2] = 0$ $\mathcal{C} = P_2^2 + P_1^2$ $H_{(1)} = SO(2)$ $H_{(2)} = \mathbb{R}$ $H_{(02)} = \mathbb{R}$	Hyperbolic $SO(2,1)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = -P_1$ $[P_1, P_2] = -J_{12}$ $\mathcal{C} = P_2^2 + P_1^2 - J_{12}^2$ $H_{(1)} = SO(2)$ $H_{(2)} = SO(1,1)$ $H_{(02)} = SO(1,1)$
Parabolic $\kappa_2 = 0$	Co-Euclidean oscillating NH $ISO(2)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = 0$ $[P_1, P_2] = J_{12}$ $\mathcal{C} = P_2^2 + J_{12}^2$ $H_{(1)} = \mathbb{R}$ $H_{(2)} = SO(2)$ $H_{(02)} = \mathbb{R}$	Galilean $IISO(1)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = 0$ $[P_1, P_2] = 0$ $\mathcal{C} = P_2^2$ $H_{(1)} = \mathbb{R}$ $H_{(2)} = \mathbb{R}$ $H_{(02)} = \mathbb{R}$	Co-Minkowskian expanding NH $ISO(1,1)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = 0$ $[P_1, P_2] = -J_{12}$ $\mathcal{C} = P_2^2 - J_{12}^2$ $H_{(1)} = \mathbb{R}$ $H_{(2)} = SO(1,1)$ $H_{(02)} = \mathbb{R}$
Hyperbolic $\kappa_2 = -1$	Co-hyperbolic anti-de Sitter $SO(2,1)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = P_1$ $[P_1, P_2] = J_{12}$ $\mathcal{C} = P_2^2 - P_1^2 + J_{12}^2$ $H_{(1)} = SO(1,1)$ $H_{(2)} = SO(2)$ $H_{(02)} = SO(1,1)$	Minkowskian $ISO(1,1)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = P_1$ $[P_1, P_2] = 0$ $\mathcal{C} = P_2^2 - P_1^2$ $H_{(1)} = SO(1,1)$ $H_{(2)} = \mathbb{R}$ $H_{(02)} = \mathbb{R}$	Doubly hyperbolic de Sitter $SO(2,1)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = P_1$ $[P_1, P_2] = -J_{12}$ $\mathcal{C} = P_2^2 - P_1^2 - J_{12}^2$ $H_{(1)} = SO(1,1)$ $H_{(2)} = SO(1,1)$ $H_{(02)} = SO(2)$

- $S^2_{[\kappa_1],\kappa_2}$ is a (1 + 1)D spacetime, and points in $S^2_{[\kappa_1],\kappa_2}$ are *spacetime events*; the spacetime curvature equals κ_1 and is related to the usual universe (time) radius τ by $\kappa_1 = \pm 1/\tau^2$.
- The space of first-kind lines $S^2_{\kappa_1,[\kappa_2]}$ corresponds to the space of *timelike lines*. The coefficient κ_2 is the curvature of the space of timelike lines, linked to the relativistic constant c as $\kappa_2 = -1/c^2$. Relativistic spacetimes occur for $\kappa_2 < 0$ (the signature of the metric is $\mathrm{diag}(1, -1/c^2)$) and their non-relativistic limits correspond to $\kappa_2 = 0$.
- The space of second-kind lines $SO_{\kappa_1,\kappa_2}(3)/H_{(02)}$ is the 2D space of *spacelike lines*.

We have three homogeneous 'absolute-time' spacetimes for $\kappa_2 = 0$: oscillating NH for $\kappa_1 > 0$, Galilean for $\kappa_1 = 0$ and expanding NH for $\kappa_1 < 0$; they are degenerate Riemannian spacetimes with constant curvature κ_1 and a degenerate ('absolute-time') metric, which is the $c = \infty$ limit of the time metric in relativity. For $\kappa_2 = -1/c^2 < 0$ we find three 'relative-time' spacetimes: anti-de Sitter ($\kappa_1 > 0$), Minkowskian ($\kappa_1 = 0$) and de Sitter ($\kappa_1 < 0$); these are pseudo-Riemannian spacetimes with a metric of Lorentzian type and constant curvature κ_1.

The limits $\kappa_1 \to 0 \equiv \tau \to \infty$ and $\kappa_2 \to 0 \equiv c \to \infty$ correspond to a spacetime contraction and a speed–space contraction, respectively.

The three remaining geometries with $\kappa_2 > 0$ do not admit such a kinematic interpretation. They are the well known Riemannian spaces with constant curvature κ_1. In these cases, the sets of first- and second-kind lines coincide, because in these cases (and only these) the generators P_1 and P_2 are conjugated. This is why only these three spaces fulfil the usual definition of two-point homogeneity; as we show in this paper there is no compelling reason to restrict any joint study only to these three cases.

2.2. *Matrix realization of the Cayley–Klein groups and the vector model of Cayley–Klein spaces*

The following 3D real matrix representation of the CK algebra $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$

$$P_1 = \begin{pmatrix} 0 & -\kappa_1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 0 & 0 & -\kappa_1\kappa_2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad J_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\kappa_2 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.9}$$

gives rise to a natural realization of the CK group $SO_{\kappa_1,\kappa_2}(3)$ as a group of linear transformations in an ambient linear space $\mathbb{R}^3 = (x^0, x^1, x^2)$ in which $SO_{\kappa_1,\kappa_2}(3)$ acts as the group of linear isometries of a bilinear form with matrix: $\Lambda = \mathrm{diag}(1, \kappa_1, \kappa_1\kappa_2)$. The exponential of the matrices (2.9) leads to a representation of the one-parameter subgroups $H_{(2)}$, $H_{(02)}$ and $H_{(1)}$ generated by P_1, P_2 and J_{12} as

$$\begin{aligned} \exp(\alpha P_1) &= \begin{pmatrix} C_{\kappa_1}(\alpha) & -\kappa_1 S_{\kappa_1}(\alpha) & 0 \\ S_{\kappa_1}(\alpha) & C_{\kappa_1}(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \exp(\beta P_2) &= \begin{pmatrix} C_{\kappa_1\kappa_2}(\beta) & 0 & -\kappa_1\kappa_2 S_{\kappa_1\kappa_2}(\beta) \\ 0 & 1 & 0 \\ S_{\kappa_1\kappa_2}(\beta) & 0 & C_{\kappa_1\kappa_2}(\beta) \end{pmatrix} \\ \exp(\gamma J_{12}) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{\kappa_2}(\gamma) & -\kappa_2 S_{\kappa_2}(\gamma) \\ 0 & S_{\kappa_2}(\gamma) & C_{\kappa_2}(\gamma) \end{pmatrix} \end{aligned} \tag{2.10}$$

where the generalized cosine $C_\kappa(x)$ and sine $S_\kappa(x)$ functions are defined by [9–11]

$$C_\kappa(x) := \sum_{l=0}^{\infty} (-\kappa)^l \frac{x^{2l}}{(2l)!} = \begin{cases} \cos\sqrt{\kappa}\, x & \kappa > 0 \\ 1 & \kappa = 0 \\ \cosh\sqrt{-\kappa}\, x & \kappa < 0 \end{cases} \tag{2.11}$$

$$S_\kappa(x) := \sum_{l=0}^{\infty} (-\kappa)^l \frac{x^{2l+1}}{(2l+1)!} = \begin{cases} \dfrac{1}{\sqrt{\kappa}} \sin\sqrt{\kappa}\, x & \kappa > 0 \\ x & \kappa = 0 \\ \dfrac{1}{\sqrt{-\kappa}} \sinh\sqrt{-\kappa}\, x & \kappa < 0. \end{cases} \tag{2.12}$$

Two other useful functions are the 'versed sine' $V_\kappa(x)$ and the tangent $T_\kappa(x)$:

$$V_\kappa(x) := \frac{1}{\kappa}(1 - C_\kappa(x)) \qquad T_\kappa(x) := \frac{S_\kappa(x)}{C_\kappa(x)}. \tag{2.13}$$

These curvature-dependent trigonometric functions coincide with the usual circular and hyperbolic ones for $\kappa = 1$ and -1, respectively; the case $\kappa = 0$ provides the parabolic or Galilean functions: $C_0(x) = 1$, $S_0(x) = x$ and $V_0(x) = x^2/2$. Several identities for these functions, which are necessary for further development, are included in the appendix.

The action of $SO_{\kappa_1,\kappa_2}(3)$ on $\mathbb{R}^3$ is linear but not transitive, since it conserves the quadratic form $(x^0)^2 + \kappa_1(x^1)^2 + \kappa_1\kappa_2(x^2)^2$, and the subgroup $H_{(1)}$ is the isotropy subgroup of the point $O \equiv (1, 0, 0)$, that is, the origin in the space $S^2_{[\kappa_1],\kappa_2}$. The action becomes transitive on the orbit in $\mathbb{R}^3$ of the point O, which is contained in the 'sphere' Σ:

$$\Sigma \equiv (x^0)^2 + \kappa_1(x^1)^2 + \kappa_1\kappa_2(x^2)^2 = 1. \tag{2.14}$$

This orbit can be identified with the space of points $S^2_{[\kappa_1],\kappa_2} \equiv SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_2}(2)$ of the CK geometry and the coordinates (x^0, x^1, x^2) are the *Weierstrass coordinates*, while $(x^1/x^0, x^2/x^0)$ are the *Beltrami coordinates*. The induced metric on the CK sphere Σ should be defined as the quotient by κ_1 of the restriction of the flat ambient metric $\mathrm{d}l^2 = (\mathrm{d}x^0)^2 + \kappa_1(\mathrm{d}x^1)^2 + \kappa_1\kappa_2(\mathrm{d}x^2)^2$; this is always well defined because the restriction of the flat metric $\mathrm{d}l^2$ to the CK sphere contains κ_1 as a factor [16].

3. The compatibility conditions for a triangular loop

We now come to the main objective of this paper, which is the study of the trigonometry of the nine CK spaces introduced in the previous section.

In the Euclidean plane three points always determine a triangle unambiguously: any two points are connected by a single geodesic segment, and all triangles are of the same type. In the sphere three points do not determine a single triangle, because two generic points can be joined by two segments on the same geodesic, yet all triangles are of the same type. In the Minkowskian plane two points can always be joined by a single geodesic segment (as in the Euclidean plane), but this segment can be of timelike, spacelike and isotropic type, so here three points do determine a single triangle, but not all triangles are of the same type. Finally, in the anti-de Sitter spacetime both complications may appear together: there are three types of sides, and two points with timelike separation can be joined, as in the sphere, by two different timelike geodesic segments.

To avoid unnecessary complications, it is better to introduce the concept of the *triangular loop*, which affords a well defined replacement of the imprecise idea of a 'triangle as three points'. A triangular loop can be considered either as a triangular point loop or as a triangular line loop, and we will need simultaneous consideration of both aspects. Furthermore, and according to the type of the 'sides', there are several different types of triangular loops, which merge into a single type in the Riemannian case $\kappa_2 > 0$. Henceforth, we will deal exclusively with *first-kind* triangular loops (i.e. timelike in the kinematic spaces) so that hereafter we can omit the reference to the first-kind type of all lines.

A *triangular point loop* can be considered as two different paths for a point going from an initial position C to a final one B. One path will be the direct one along the segment of line a determined by C and B. The other will be a two-step path constructed from two segments of lines going from C to an intermediary point A along a line b and then from A to B along line c (see figure 1(a)). For most purposes it is better to look at the triangular point loop as a single (possibly open) polygonal curve, obtained from line a by replacing the segment CB

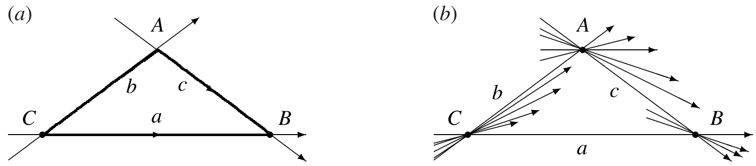

Figure 1. (*a*) Triangular point loop. (*b*) Triangular line loop.

by the two geodesic segments CA and AB; this curve will be considered as an oriented and cooriented curve, and will be only closed when the geodesic a itself is closed. This view can be dualized, and the loop can also be considered as a *triangular line loop*: the dual of the single curve associated with the point loop is a moving line which starts at a, then rotates around C going to b, then around A towards c and finally comes back to a by means of a rotation around B (see figure 1(*b*)).

Each side of the loop determines the generator of *translations* along the side, up to a non-zero scale factor which should be split into a sign (corresponding to one of the two possible orientations of the line) and a positive scale factor (corresponding to the choice of a unit length). The restriction to first-kind sides means that:

P1 The three generators P_a, P_b, P_c are either equal or opposite to some conjugate to the single fiducial generator of translations P_1 along first-kind lines.

We shall now perform a fiducial choice of the still undetermined factors in these generators, which we will henceforth assume to be fixed, according to a second condition.

P2 The positive sense of translation generated by P_a, P_b, P_c agrees with the orientation for the point loop as a single curve.

The meaning of these conditions can be appreciated more clearly for the kinematic geometries with $\kappa_2 \leqslant 0$, for which the first condition embodies the timelike character of the three lines (here $P_1 \equiv \mathcal{H}$ generates the future time translation along the fiducial timelike line), and the second condition corresponds to the future character of a timelike line loop. In the Riemannian cases ($\kappa_2 > 0$) all geodesics can be considered to be simultaneously of both first and second kind; then the first condition is automatic, while the second one can always be clearly fulfilled. The important fact is that for any triangular loop, a choice of P_a, P_b, P_c satisfying these two conditions is always possible, and this is so simultaneously for the *nine* CK geometries.

On the dual hand, the generators J_A, J_B, J_C of *rotations* around the vertices $A \equiv b \cap c$, $B \equiv c \cap a$, $C \equiv a \cap b$ are again determined up to sign and a positive scale factor, which we shall choose so as to satisfy two conditions, dual to the previous ones.

J1 The three generators J_A, J_B, J_C are conjugated by means of some group transformation to the single fiducial generator of rotations J_{12}.

J2 The positive sense of rotation around each vertex is the correct one determined by the given orientation and co-orientation of the loop as a curve.

Now, for a given triangular loop, we denote by a, b, c the three side lengths (which are positive and unoriented distances), B, C the two inner angles and A an external angle (see figure 2). In the kinematic cases, the lengths a, b, c will be the proper times along the sides between their end events, and the angles A, B, C are the relative rapidities between the timelike lines at each vertex; the triangle loop, seen as a single curve, is the worldline of the travelling twin in the twin pseudo-paradox.

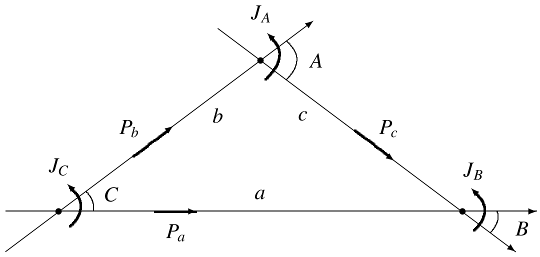

Figure 2. Triangle with three first-kind sides a, b, c, two inner angles B, C and an external angle A.

The generators P_a, P_b, P_c; J_A, J_B, J_C are not independent. They are related by several *compatibility conditions*:

$$\begin{aligned} P_b &= \mathrm{e}^{CJ_C} P_a \mathrm{e}^{-CJ_C} \qquad & J_B &= \mathrm{e}^{cP_c} J_A \mathrm{e}^{-cP_c} \\ P_c &= \mathrm{e}^{-AJ_A} P_b \mathrm{e}^{AJ_A} \qquad & J_C &= \mathrm{e}^{-aP_a} J_B \mathrm{e}^{aP_a} \\ P_a &= \mathrm{e}^{BJ_B} P_c \mathrm{e}^{-BJ_B} \qquad & J_A &= \mathrm{e}^{bP_b} J_C \mathrm{e}^{-bP_b} \end{aligned} \tag{3.1}$$

which can be considered as giving an implicit group-theoretical definition for the three sides and the three angles. Our main contention is that all the trigonometry of the space is *completely* contained in these equations, which have as a remarkable property their explicit *duality* (due to $\mathcal{D}$ (2.7)) under the interchange $a, b, c \leftrightarrow A, B, C$ and $P \leftrightarrow J$.

The first equation in (3.1) gives the translation generator P_b as a conjugate of P_a by means of a rotation around C; the same equation read inversely gives P_a as a conjugate of P_b by means of the inverse rotation around C. We will refer to them as $P_b(P_a)$ or $P_a(P_b)$; likewise the remaining equations will be referred to as $P_c(P_b)$, $P_a(P_c)$, $J_B(J_A)$, etc.

By cyclic substitution in the three equations $P_a(P_c)$, $P_c(P_b)$ and $P_b(P_a)$ we find

$$\mathrm{e}^{BJ_B}\mathrm{e}^{-AJ_A}\mathrm{e}^{CJ_C} P_a \mathrm{e}^{-CJ_C}\mathrm{e}^{AJ_A}\mathrm{e}^{-BJ_B} = P_a. \tag{3.2}$$

Likewise, a dual parallel process allows us to derive an identity involving J_C:

$$\mathrm{e}^{-aP_a}\mathrm{e}^{cP_c}\mathrm{e}^{bP_b} J_C \mathrm{e}^{-bP_b}\mathrm{e}^{-cP_c}\mathrm{e}^{aP_a} = J_C. \tag{3.3}$$

Equations (3.2) and (3.3) can be written alternatively as

$$\begin{aligned} &\mathrm{e}^{BJ_B}\mathrm{e}^{-AJ_A}\mathrm{e}^{CJ_C} \text{ must commute with } P_a \\ &\mathrm{e}^{-aP_a}\mathrm{e}^{cP_c}\mathrm{e}^{bP_b} \text{ must commute with } J_C. \end{aligned} \tag{3.4}$$

3.1. Loop excesses and loop equations

The content of (3.4) means that the product $\mathrm{e}^{-aP_a}\mathrm{e}^{cP_c}\mathrm{e}^{bP_b}$ of the three translations along the three sides of the triangle $C \xrightarrow{b} A \xrightarrow{c} B \xrightarrow{-a} C$ moves the base point C along the triangle and returns it back to its original position, so it must necessarily be a *rotation* around the vertex C by some angle $-\Delta_C$ and must commute with J_C:

$$\mathrm{e}^{-aP_a}\mathrm{e}^{cP_c}\mathrm{e}^{bP_b} = \mathrm{e}^{-\Delta_C J_C}. \tag{3.5}$$

Likewise, the product $\mathrm{e}^{BJ_B}\mathrm{e}^{-AJ_A}\mathrm{e}^{CJ_C}$ of the three rotations around the three vertices, $a \xrightarrow{C} b \xrightarrow{-A} c \xrightarrow{B} a$ must be a translation along the side a by an amount $-\delta_a$:

$$\mathrm{e}^{BJ_B}\mathrm{e}^{-AJ_A}\mathrm{e}^{CJ_C} = \mathrm{e}^{-\delta_a P_a}. \tag{3.6}$$

The two quantities δ_a and Δ_C are so far unknown. To find them we start with the equation $P_c(P_b)$ in (3.1), replace P_c by $\mathrm{e}^{-cP_c}P_c\mathrm{e}^{cP_c}$ and then substitute $P_c(P_a)$ to obtain

$$\mathrm{e}^{-AJ_A}P_b\mathrm{e}^{AJ_A} = \mathrm{e}^{-cP_c}\mathrm{e}^{-BJ_B}P_a\mathrm{e}^{BJ_B}\mathrm{e}^{cP_c}. \tag{3.7}$$

We introduce $J_B(J_C)$ and simplify trivially to find

$$\mathrm{e}^{-AJ_A}P_b\mathrm{e}^{AJ_A} = \mathrm{e}^{-cP_c}\mathrm{e}^{aP_a}\mathrm{e}^{-BJ_C}P_a\mathrm{e}^{BJ_C}\mathrm{e}^{-aP_a}\mathrm{e}^{cP_c} \tag{3.8}$$

which is equivalent to

$$\mathrm{e}^{BJ_C}\mathrm{e}^{-aP_a}\mathrm{e}^{cP_c}\mathrm{e}^{-AJ_A}P_b\mathrm{e}^{AJ_A}\mathrm{e}^{-cP_c}\mathrm{e}^{aP_a}\mathrm{e}^{-BJ_C} = P_a. \tag{3.9}$$

Now we use $J_A(J_C)$, simplify and finally substitute $P_b(P_a)$. This gives

$$\mathrm{e}^{BJ_C}\mathrm{e}^{-aP_a}\mathrm{e}^{cP_c}\mathrm{e}^{bP_b}\mathrm{e}^{-AJ_C}\mathrm{e}^{CJ_C}P_a\mathrm{e}^{-CJ_C}\mathrm{e}^{AJ_C}\mathrm{e}^{-bP_b}\mathrm{e}^{-cP_c}\mathrm{e}^{aP_a}\mathrm{e}^{-BJ_C} = P_a. \tag{3.10}$$

Due to (3.4) we can write

$$\mathrm{e}^{-aP_a}\mathrm{e}^{cP_c}\mathrm{e}^{bP_b}\mathrm{e}^{(-A+B+C)J_C}P_a\mathrm{e}^{-(-A+B+C)J_C}\mathrm{e}^{-bP_b}\mathrm{e}^{-cP_c}\mathrm{e}^{aP_a} = P_a \tag{3.11}$$

$$\mathrm{e}^{-aP_a}\mathrm{e}^{cP_c}\mathrm{e}^{bP_b}\mathrm{e}^{(-A+B+C)J_C} \text{ must commute with } P_a. \tag{3.12}$$

And by taking into account (3.4) and (3.12), we immediately conclude that

$$\mathrm{e}^{-aP_a}\mathrm{e}^{cP_c}\mathrm{e}^{bP_b}\mathrm{e}^{(-A+B+C)J_C} = 1 \tag{3.13}$$

since the identity is the only element of $SO_{\kappa_1,\kappa_2}(3)$ that commutes with two such generators as P_a and J_C. This equation can also be written as

$$\mathrm{e}^{-aP_a}\mathrm{e}^{cP_c}\mathrm{e}^{bP_b} = \mathrm{e}^{-(-A+B+C)J_C} \tag{3.14}$$

and so it gives the unknown angle $\Delta_C = -A+B+C$ appearing in (3.5). A very similar procedure allows us to derive two analogous equations for the quantities Δ_A, Δ_B:

$$\begin{aligned} \mathrm{e}^{bP_b}\mathrm{e}^{-aP_a}\mathrm{e}^{cP_c} &= \mathrm{e}^{-(-A+B+C)J_A} \\ \mathrm{e}^{cP_c}\mathrm{e}^{bP_b}\mathrm{e}^{-aP_a} &= \mathrm{e}^{-(-A+B+C)J_B} \end{aligned} \tag{3.15}$$

hence we obtain

$$\Delta_A = \Delta_B = \Delta_C = -A+B+C \equiv \Delta. \tag{3.16}$$

The quantity Δ will be called the *angular excess* of the triangle loop, and fits into the view of the point loop as a single curve which starts on the geodesic a, and successively rotates by angles C, $-A$ and B around the three vertices of the triangle, so that $-A+B+C$ should be seen as the (oriented) total angle turned by the line loop. Equations (3.14) and (3.15), to be called the *point loop equations*, simply state that the product of the three translations along the oriented sides of the triangle loop equals a rotation around the base point of the loop, with an angle equal to minus the *angular excess* of the triangle loop.

Duality implies that the dual partners of equations (3.14) and (3.15) given by

$$\begin{aligned} \mathrm{e}^{-AJ_A}\mathrm{e}^{CJ_C}\mathrm{e}^{BJ_B} &= \mathrm{e}^{-(-a+b+c)P_c} \\ \mathrm{e}^{BJ_B}\mathrm{e}^{-AJ_A}\mathrm{e}^{CJ_C} &= \mathrm{e}^{-(-a+b+c)P_a} \\ \mathrm{e}^{CJ_C}\mathrm{e}^{BJ_B}\mathrm{e}^{-AJ_A} &= \mathrm{e}^{-(-a+b+c)P_b} \end{aligned} \tag{3.17}$$

also hold, so that

$$\delta_a = \delta_b = \delta_c = -a+b+c \equiv \delta \tag{3.18}$$

which will be called the *lateral excess* of the triangle. This appears as the (oriented) total length of the point loop, where b and c are traversed in the same sense as the orientations chosen for P_b, P_c, but a is traversed backwards relative to P_a. Therefore, equation (3.17), to be called *line loop equations*, gives the product of the three oriented rotations around the three vertices of a triangle as a translation along the base line of the loop, by an amount equal to minus the *lateral excess* of the triangle loop.

Consequently, the canonical parameters of the 'holonomy' rotation, or of the dual 'holonomy' translation are independent of the base point or line, and are therefore associated with the triangle loop in an intrinsic way. As we will see shortly, the excesses Δ and δ are directly related to other natural quantities, the area and co-area of the triangular loop.

3.2. The basic trigonometric identity

Potentially, each of equations (3.14), (3.15) and (3.17) contains all the trigonometry of any CK space. However, sides and angles appear in these equations not only explicitly as canonical parameters, but also implicitly hidden inside the translation and rotation generators. This prompts the search for another relation, equivalent to the previous ones but more suitable for displaying the trigonometric equations; this new equation is indeed the bridge between the former equations and the trigonometry of the space. The main idea is to express *all* the generators as suitable conjugates of *one* translation generator and *one* rotation generator, which we will take as primitive *independent* generators.

A natural choice is to take P_a and J_C as 'basic' independent generators. Next, by using the compatibility conditions (3.1) we *define* the remaining triangle generators P_b, J_A, P_c, J_B in terms of the previous ones. After full expansion and simplification we obtain that

$$\begin{aligned} P_b &:= \mathrm{e}^{CJ_C} P_a \mathrm{e}^{-CJ_C} \\ J_A &:= \mathrm{e}^{CJ_C}\mathrm{e}^{bP_a} J_C \mathrm{e}^{-bP_a}\mathrm{e}^{-CJ_C} \\ P_c &:= \mathrm{e}^{CJ_C}\mathrm{e}^{bP_a}\mathrm{e}^{-AJ_C} P_a \mathrm{e}^{AJ_C}\mathrm{e}^{-bP_a}\mathrm{e}^{-CJ_C} \\ J_B &:= \mathrm{e}^{CJ_C}\mathrm{e}^{bP_a}\mathrm{e}^{-AJ_C}\mathrm{e}^{cP_a} J_C \mathrm{e}^{-cP_a}\mathrm{e}^{AJ_C}\mathrm{e}^{-bP_a}\mathrm{e}^{-CJ_C}. \end{aligned} \tag{3.19}$$

By direct substitution in equation (3.14) and after obvious cancellations we find

$$\mathrm{e}^{-aP_a}\mathrm{e}^{CJ_C}\mathrm{e}^{bP_a}\mathrm{e}^{-AJ_C}\mathrm{e}^{cP_a}\mathrm{e}^{BJ_C} = 1. \tag{3.20}$$

The same process starting from any of equations (3.15) or (3.17) leads again to (3.20) with the terms cyclically permuted. This justifies calling equation (3.20) the *basic trigonometric equation*; it is clearly a self-dual equation.

The results obtained so far can be summed up as follows.

Theorem 1. *Sides a, b, c and angles A, B, C of any triangle loop are linked by the single self-dual group identity called the* basic trigonometric identity

$$\mathrm{e}^{-aP}\mathrm{e}^{CJ}\mathrm{e}^{bP}\mathrm{e}^{-AJ}\mathrm{e}^{cP}\mathrm{e}^{BJ} = 1 \tag{3.21}$$

where P, J are the generators of translations along any *fixed fiducial line l, and of rotations around any fixed fiducial point O on the line l.*

Theorem 2. *Let P_a, P_b, P_c be the generators of* translations *along the three sides of a triangle (whose lengths are a, b, c), and J_A, J_B, J_C the generators of* rotations *around the* *three vertices (with angles A, B, C). Then we have two sets of identities, called the* point loop *and the* line loop *equations for the triangle:*

$$\begin{aligned} &e^{bP_b}e^{-aP_a}e^{cP_c} = e^{-(-A+B+C)J_A} && e^{BJ_B}e^{-AJ_A}e^{CJ_C} = e^{-(-a+b+c)P_a} \\ &e^{cP_c}e^{bP_b}e^{-aP_a} = e^{-(-A+B+C)J_B} && e^{CJ_C}e^{BJ_B}e^{-AJ_A} = e^{-(-a+b+c)P_b} \\ &e^{-aP_a}e^{cP_c}e^{bP_b} = e^{-(-A+B+C)J_C} && e^{-AJ_A}e^{CJ_C}e^{BJ_B} = e^{-(-a+b+c)P_c}. \end{aligned} \tag{3.22}$$

Furthermore, each of these identities is equivalent to the identity in theorem 1.

We emphasize that all of these equations hold in the same explicit form for *all* 2D real CK geometries, as no *explicit* κ_1, κ_2 ever appear in them.

4. Equations of trigonometry in the nine Cayley–Klein spaces

The most convenient way to obtain the trigonometric equations of the space $S^2_{[\kappa_1],\kappa_2}$ is to start with the basic trigonometric identity (3.21), in which from now on the two generators P and J will be taken to be exactly P_1 and J_{12}. For convenience, we write (3.21) as

$$e^{-aP}e^{CJ}e^{bP} = e^{-BJ}e^{-cP}e^{AJ}. \tag{4.1}$$

By considering this identity in the fundamental 3D vector representation of the motion group (2.10) we obtain an equality between 3×3 matrices, giving rise to nine identities:

$$\begin{array}{ll} \text{1c} & C_{\kappa_1}(c) = C_{\kappa_1}(a)C_{\kappa_1}(b) + \kappa_1 S_{\kappa_1}(a)S_{\kappa_1}(b)C_{\kappa_2}(C) \\ \text{1C} & C_{\kappa_2}(C) = C_{\kappa_2}(A)C_{\kappa_2}(B) + \kappa_2 S_{\kappa_2}(A)S_{\kappa_2}(B)C_{\kappa_1}(c) \\ \text{2cA} \equiv \text{2aC} & S_{\kappa_1}(c)S_{\kappa_2}(A) = S_{\kappa_1}(a)S_{\kappa_2}(C) \\ \text{2cB} \equiv \text{2bC} & S_{\kappa_1}(c)S_{\kappa_2}(B) = S_{\kappa_1}(b)S_{\kappa_2}(C) \\ \text{3cA} & S_{\kappa_1}(c)C_{\kappa_2}(A) = -C_{\kappa_1}(a)S_{\kappa_1}(b) + S_{\kappa_1}(a)C_{\kappa_1}(b)C_{\kappa_2}(C) \\ \text{3cB} & S_{\kappa_1}(c)C_{\kappa_2}(B) = C_{\kappa_1}(b)S_{\kappa_1}(a) - S_{\kappa_1}(b)C_{\kappa_1}(a)C_{\kappa_2}(C) \\ \text{3Ca} & S_{\kappa_2}(C)C_{\kappa_1}(a) = -C_{\kappa_2}(A)S_{\kappa_2}(B) + S_{\kappa_2}(A)C_{\kappa_2}(B)C_{\kappa_1}(c) \\ \text{3Cb} & S_{\kappa_2}(C)C_{\kappa_1}(b) = C_{\kappa_2}(B)S_{\kappa_2}(A) - S_{\kappa_2}(B)C_{\kappa_2}(A)C_{\kappa_1}(c) \\ \text{4AB} \equiv \text{4ab} & \kappa_2 S_{\kappa_2}(A)S_{\kappa_2}(B) + C_{\kappa_2}(A)C_{\kappa_2}(B)C_{\kappa_1}(c) \\ & \qquad = \kappa_1 S_{\kappa_1}(a)S_{\kappa_1}(b) + C_{\kappa_1}(a)C_{\kappa_1}(b)C_{\kappa_2}(C). \end{array} \tag{4.2}$$

The tag assigned to each equation is self-descriptive: all equations are either self-dual (for instance 2cA $\equiv$ 2aC) or appear in mutually dual pairs (as (1c, 1C) or (3cA, 3Ca)). We remark that all sides (respectively, angles) appear in the equations through the trigonometric functions which have κ_1 (respectively, κ_2) as a label.

To describe clearly the structure and dependence between these equations, it is better to consider the group of equations (4.2) together with two similar groups, each equivalent as a set of equations to the previous one. These can be obtained by starting from the basic identity written in two alternative forms as

$$e^{bP}e^{-AJ}e^{cP} = e^{-CJ}e^{aP}e^{-BJ} \qquad e^{cP}e^{BJ}e^{-aP} = e^{AJ}e^{-bP}e^{-CJ}. \tag{4.3}$$

By writing them in the fundamental representation (2.10), we obtain two other sets of equations very similar to (4.2). All the equations taken altogether can be grouped into the following:

- Three *cosine theorems* for sides:

$$\begin{array}{ll} \text{1a} & C_{\kappa_1}(a) = C_{\kappa_1}(b)C_{\kappa_1}(c) - \kappa_1 S_{\kappa_1}(b)S_{\kappa_1}(c)C_{\kappa_2}(A) \\ \text{1b} & C_{\kappa_1}(b) = C_{\kappa_1}(a)C_{\kappa_1}(c) + \kappa_1 S_{\kappa_1}(a)S_{\kappa_1}(c)C_{\kappa_2}(B) \\ \text{1c} & C_{\kappa_1}(c) = C_{\kappa_1}(a)C_{\kappa_1}(b) + \kappa_1 S_{\kappa_1}(a)S_{\kappa_1}(b)C_{\kappa_2}(C) \end{array} \tag{4.4}$$

and three *dual cosine theorems* for angles:

$$\begin{array}{ll} \text{1A} & C_{\kappa_2}(A) = C_{\kappa_2}(B)C_{\kappa_2}(C) - \kappa_2 S_{\kappa_2}(B)S_{\kappa_2}(C)C_{\kappa_1}(a) \\ \text{1B} & C_{\kappa_2}(B) = C_{\kappa_2}(A)C_{\kappa_2}(C) + \kappa_2 S_{\kappa_2}(A)S_{\kappa_2}(C)C_{\kappa_1}(b) \\ \text{1C} & C_{\kappa_2}(C) = C_{\kappa_2}(A)C_{\kappa_2}(B) + \kappa_2 S_{\kappa_2}(A)S_{\kappa_2}(B)C_{\kappa_1}(c). \end{array} \tag{4.5}$$

- One self-dual *sine theorem* (obtained from the six equations 2cA, 2cB, ...):

$$2 \qquad \frac{S_{\kappa_1}(a)}{S_{\kappa_2}(A)} = \frac{S_{\kappa_1}(b)}{S_{\kappa_2}(B)} = \frac{S_{\kappa_1}(c)}{S_{\kappa_2}(C)}. \tag{4.6}$$

- Six 'side addition' theorems which correspond to the tags 3aB, 3aC, 3bA, 3bC, 3cA, 3cB and six dual 'angle addition' theorems 3Ab, 3Ac, 3Ba, 3Bc, 3Ca, 3Cb.
- Three self-dual theorems 4AB $\equiv$ 4ab, 4AC $\equiv$ 4ac, 4BC $\equiv$ 4bc.

Note the sign differences in 1a (1A) as compared with 1b, 1c (1B, 1C). These equations are a *complete* set of trigonometric equations for any values of the constants κ_1, κ_2, but most of them reduce to simpler, or even trivial ones in the degenerate cases $\kappa_i = 0$. In particular, the cosine theorems (4.4) (respectively, (4.5)) give rise to trivial identities $1 = 1$ when $\kappa_1 = 0$ (respectively, $\kappa_2 = 0$). This can be circumvented by writing these equations in an alternative form. Take the cosine equation for the side c in the generic case with $\kappa_1 \neq 0$, and write all cosines in terms of versed sines by introducing (A.2):

$$1 - \kappa_1 V_{\kappa_1}(c) = (1 - \kappa_1 V_{\kappa_1}(a))(1 - \kappa_1 V_{\kappa_1}(b)) + \kappa_1 S_{\kappa_1}(a)S_{\kappa_1}(b)(1 - \kappa_2 V_{\kappa_2}(C)). \tag{4.7}$$

By expanding and cancelling a common factor κ_1 we find that

$$V_{\kappa_1}(c) = V_{\kappa_1}(a) + V_{\kappa_1}(b) - \kappa_1 V_{\kappa_1}(a)V_{\kappa_1}(b) - S_{\kappa_1}(a)S_{\kappa_1}(b) + \kappa_2 S_{\kappa_1}(a)S_{\kappa_1}(b)V_{\kappa_2}(C) \tag{4.8}$$

which is rather simplified by means of (A.10). The remaining cosine theorems allow a similar reformulation. Thus we obtain the following alternative expressions:

$$\begin{array}{ll} 1'\text{a} & V_{\kappa_1}(a) - V_{\kappa_1}(b+c) = -\kappa_2 S_{\kappa_1}(b)S_{\kappa_1}(c)V_{\kappa_2}(A) \\ 1'\text{b} & V_{\kappa_1}(b) - V_{\kappa_1}(a-c) = \kappa_2 S_{\kappa_1}(a)S_{\kappa_1}(c)V_{\kappa_2}(B) \\ 1'\text{c} & V_{\kappa_1}(c) - V_{\kappa_1}(a-b) = \kappa_2 S_{\kappa_1}(a)S_{\kappa_1}(b)V_{\kappa_2}(C) \end{array} \tag{4.9}$$

$$\begin{array}{ll} 1'\text{A} & V_{\kappa_2}(A) - V_{\kappa_2}(B+C) = -\kappa_1 S_{\kappa_2}(B)S_{\kappa_2}(C)V_{\kappa_1}(a) \\ 1'\text{B} & V_{\kappa_2}(B) - V_{\kappa_2}(A-C) = \kappa_1 S_{\kappa_2}(A)S_{\kappa_2}(C)V_{\kappa_1}(b) \\ 1'\text{C} & V_{\kappa_2}(C) - V_{\kappa_2}(A-B) = \kappa_1 S_{\kappa_2}(A)S_{\kappa_2}(B)V_{\kappa_1}(c). \end{array} \tag{4.10}$$

(Note again the sign difference in 1′a and 1′A.) These relations are clearly equivalent to (4.4) (respectively, (4.5)) when $\kappa_1 \neq 0$ (respectively, $\kappa_2 \neq 0$), but do not reduce to trivial identities when $\kappa_1 = 0$ or $\kappa_2 = 0$. In this sense they can be considered as the 'good' form of cosine and dual cosine equations. Equations (4.9) and (4.10) still allow another alternative very useful form. Consider the half sums of the three sides and of the three angles (cf (A.15)):

$$p = \tfrac{1}{2}(a+b+c) \qquad P = \tfrac{1}{2}(A+B+C). \tag{4.11}$$

By introducing the identities (A.24) and (A.25) applied to the three sides a, b, c and angles A, B, C into the cosine theorems (4.9) and (4.10), we obtain

$$\begin{array}{ll} 1''\mathrm{a} & 2S_{\kappa_1}(p-a)S_{\kappa_1}(p) = \kappa_2 S_{\kappa_1}(b)S_{\kappa_1}(c)V_{\kappa_2}(A) \\ 1''\mathrm{b} & 2S_{\kappa_1}(p-a)S_{\kappa_1}(p-c) = \kappa_2 S_{\kappa_1}(a)S_{\kappa_1}(c)V_{\kappa_2}(B) \\ 1''\mathrm{c} & 2S_{\kappa_1}(p-a)S_{\kappa_1}(p-b) = \kappa_2 S_{\kappa_1}(a)S_{\kappa_1}(b)V_{\kappa_2}(C) \end{array} \tag{4.12}$$

$$\begin{array}{ll} 1''\mathrm{A} & 2S_{\kappa_2}(P-A)S_{\kappa_2}(P) = \kappa_1 S_{\kappa_2}(B)S_{\kappa_2}(C)V_{\kappa_1}(a) \\ 1''\mathrm{B} & 2S_{\kappa_2}(P-A)S_{\kappa_2}(P-C) = \kappa_1 S_{\kappa_2}(A)S_{\kappa_2}(C)V_{\kappa_1}(b) \\ 1''\mathrm{C} & 2S_{\kappa_2}(P-A)S_{\kappa_2}(P-B) = \kappa_1 S_{\kappa_2}(A)S_{\kappa_2}(B)V_{\kappa_1}(c). \end{array} \tag{4.13}$$

4.1. Dependence and sets of basic equations

The equations we have obtained so far contain the whole trigonometry of the CK space $S^2_{[\kappa_1],\kappa_2}$. Nevertheless, not all of these equations can be independent: in any CK space, a triangle is completely determined by *three* independent quantities so we should expect *three* independent relations between the six quantities a, b, c; A, B, C.

Let us first discuss the case with $\kappa_1 = 0$ but $\kappa_2 \neq 0$. In these degenerate cases we obtain the well known trigonometry of the Euclidean plane ($\kappa_2 > 0$), and the less well known Lorentzian trigonometry of the (1 + 1)D Minkowskian spacetime ($\kappa_2 < 0$) [1]. The formulae (4.4) reduce to trivial identities $1 = 1$, but the alternative expressions (4.9) or (4.12) lead to the three ordinary flat Euclidean or Lorentzian cosine theorems, the latter exactly as given in [1]. For instance, 1′c gives rise to

$$\tfrac{1}{2}c^2 - \tfrac{1}{2}(a-b)^2 = \kappa_2 ab\, V_{\kappa_2}(C) \quad \equiv \quad c^2 = a^2 + b^2 - 2ab\, C_{\kappa_2}(C). \tag{4.14}$$

All the remaining equations of trigonometry do not reduce to identities and are directly meaningful, yet simpler. By taking into account the sine and cosine addition identities (A.8) and (A.9), we find that for $\kappa_1 = 0$ the content of all of the dual cosine theorems (4.5) and all of equations 3Ab, 3Ac, 3Ba, 3Bc, 3Ca, 3Cb, 4AB, 4AC, 4BC is the same and reduces to a triangular angle addition in the form

$$A = B + C \quad \equiv \quad \Delta = 0. \tag{4.15}$$

The equality $\Delta = 0$ implies that for $\kappa_1 = 0$ the holonomy (3.14) is equal to the identity, as it should in any flat space. In these flat spaces where $\kappa_1 = 0$ the angles are related by a ‘universal’ linear equation which is not dependent on the sides. This universality is why the equality $A = B + C$ is usually taken as a property of Euclidean geometry, and not as a trigonometric equation. The sine theorem (4.6) now reads

$$\frac{a}{S_{\kappa_2}(A)} = \frac{b}{S_{\kappa_2}(B)} = \frac{c}{S_{\kappa_2}(C)} \tag{4.16}$$

and the remaining equations 3aB, 3aC, 3bA, 3bC, 3cA, 3cB give each side as the sum of the projections of the other two. In particular, relations 3cA and 3cB in (4.2) reduce to

$$b = aC_{\kappa_2}(C) - cC_{\kappa_2}(A) \qquad a = bC_{\kappa_2}(C) + cC_{\kappa_2}(B). \tag{4.17}$$

Note that the three cosine theorems 1′a, 1′b, 1′c (as (4.14)) are still *independent*. From these we can derive all the remaining non-trivial equations, including the dual cosine theorem (4.15), the sine theorem (4.16) and the relations on the sum of the projections as (4.17). Henceforth, when $\kappa_1 = 0$ but $\kappa_2 \neq 0$, the canonical choice for three independent equations is the three alternative cosine theorems (4.9) for the three sides.

A fully parallel dual discussion can be repeated for the case $\kappa_2 = 0$ but with $\kappa_1 \neq 0$. Then equations (4.5) give rise to trivial identities, while (4.10) or (4.13) provide the dual cosine theorems for the angles, which are the three independent equations. The relations (4.4), 3aB, 3aC, 3bA, etc reduce to the same single equation:

$$a = b + c \quad \equiv \quad \delta = 0. \tag{4.18}$$

Finally, in the more contracted case with $\kappa_1 = \kappa_2 = 0$, that is, the (1 + 1)D Galilean geometry, the equations are fully linear and read

$$a = b + c \qquad A = B + C \qquad \frac{a}{A} = \frac{b}{B} = \frac{c}{C}. \tag{4.19}$$

All the results concerning the dependence of equations can be summed up in the following.

Theorem 3. *The full set of equations of trigonometry always contains (i.e. for any value of κ_1, κ_2) exactly* three *independent equations. Any other equation in the set is a consequence of them. According to the values of κ_1, κ_2 we find the following cases.*

- *When $\kappa_1 \neq 0$ and $\kappa_2 \neq 0$, the trigonometry follows from either (4.9) or (4.10).*
- *When $\kappa_1 = 0$ but $\kappa_2 \neq 0$, the trigonometry follows from (4.9).*
- *When $\kappa_1 \neq 0$ but $\kappa_2 = 0$, the trigonometry follows from (4.10).*
- *When $\kappa_1 = \kappa_2 = 0$, the trigonometry follows from (4.19).*

We display in table 2 the cosine, dual cosine and sine theorems for each of the nine CK geometries according to the values of the curvatures (κ_1, κ_2).

While the equations we have obtained hold for *all* the nine 2D CK geometries, the spaces whose trigonometry is well known are the three Riemannian spaces of constant curvature—the sphere, the Euclidean plane and the hyperbolic plane—which are the CK geometries with $\kappa_2 > 0$. In these three spaces the usual 'natural' choice of angle units corresponds to making $\kappa_2 = 1$. Therefore, by setting $\kappa_1 = \kappa$ and $\kappa_2 = 1$, our generic relations give directly the so-called 'absolute' form of trigonometry, which is valid for the three Riemannian spaces of constant curvature κ simultaneously [3, 7, 8]. In this connection an elementary but relevant point should be kept in mind: spherical, Euclidean and hyperbolic trigonometry is usually formulated in terms of the *inner* angles. However, when the triangle is seen as a line loop, *one* of the angles must be an external angle, like our A. When $\kappa_2 = 1$, the measure of a straight angle is equal to π, and the three internal angles α, β, γ are related to our A, B, C as $\alpha = \pi - A$, $\beta = B$, $\gamma = C$, so that the triangle angular excess Δ, usually defined as $\alpha + \beta + \gamma - \pi$ appears here as $-A + B + C$, and thus it does not involve π. These facts account for all apparent discrepancies between the particularization of the general equations given in this paper and those found in the literature for the three Riemannian spaces [27–29]. The use of the three *inner* angles makes π enter unavoidably into the angle sum, the definition of angular excess, the Gauss–Bonnet theorem, etc, and thus seems to preclude analogous relations in the cases with a locally Lorentzian metric, where π does not properly belong anymore.

Thus the choice for angles in this paper is carried out consistently in all the *nine* CK spaces and their trigonometry is formulated in a single unified way where *all* equations are analogous and directly meaningful in all cases. Hence our approach is more general than *absolute trigonometry*. Nevertheless, to unfold this view we have to abandon the implicit restriction $\kappa_2 = 1$ which amounts to measuring angles in radians and is universally enforced for the three Riemannian cases. While the curvature κ_1 allows us to explicitly distinguish

Table 2. Cosine, sine and dual cosine theorems for the nine CK spaces.

Elliptic $(+1, +1)$	Euclidean $(0, +1)$	Hyperbolic $(-1, +1)$
$SO(3)/SO(2)$	$ISO(2)/SO(2)$	$SO(2,1)/SO(2)$
$\cos a = \cos b \cos c - \sin b \sin c \cos A$	$a^2 = b^2 + c^2 + 2bc \cos A$	$\cosh a = \cosh b \cosh c + \sinh b \sinh c \cos A$
$\cos b = \cos a \cos c + \sin a \sin c \cos B$	$b^2 = a^2 + c^2 - 2ac \cos B$	$\cosh b = \cosh a \cosh c - \sinh a \sinh c \cos B$
$\cos c = \cos a \cos b + \sin a \sin b \cos C$	$c^2 = a^2 + b^2 - 2ab \cos C$	$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C$
$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$	$\frac{\sinh a}{\sin A} = \frac{\sinh b}{\sin B} = \frac{\sinh c}{\sin C}$
$\cos A = \cos B \cos C - \sin B \sin C \cos a$	$A = B + C$	$\cos A = \cos B \cos C - \sin B \sin C \cosh a$
$\cos B = \cos A \cos C + \sin A \sin C \cos b$	$B = A - C$	$\cos B = \cos A \cos C + \sin A \sin C \cosh b$
$\cos C = \cos A \cos B + \sin A \sin B \cos c$	$C = A - B$	$\cos C = \cos A \cos B + \sin A \sin B \cosh c$
Co-Euclidean $(+1, 0)$	Galilean $(0, 0)$	Co-Minkowskian $(-1, 0)$
Oscillating NH $ISO(2)/\mathbb{R}$	$IISO(1)/\mathbb{R}$	Expanding NH $ISO(1,1)/\mathbb{R}$
$a = b + c$	$a = b + c$	$a = b + c$
$b = a - c$	$b = a - c$	$b = a - c$
$c = a - b$	$c = a - b$	$c = a - b$
$\frac{\sin a}{A} = \frac{\sin b}{B} = \frac{\sin c}{C}$	$\frac{a}{A} = \frac{b}{B} = \frac{c}{C}$	$\frac{\sinh a}{A} = \frac{\sinh b}{B} = \frac{\sinh c}{C}$
$A^2 = B^2 + C^2 + 2BC \cos a$	$A = B + C$	$A^2 = B^2 + C^2 + 2BC \cosh a$
$B^2 = A^2 + C^2 - 2\,AC \cos b$	$B = A - C$	$B^2 = A^2 + C^2 - 2\,AC \cosh b$
$C^2 = A^2 + B^2 - 2\,AB \cos c$	$C = A - B$	$C^2 = A^2 + B^2 - 2\,AB \cosh c$
Co-hyperbolic $(+1, -1)$	Minkowskian $(0, -1)$	Doubly hyperbolic $(-1, -1)$
Anti-de Sitter $SO(2,1)/SO(1,1)$	$ISO(1,1)/SO(1,1)$	de Sitter $SO(2,1)/SO(1,1)$
$\cos a = \cos b \cos c - \sin b \sin c \cosh A$	$a^2 = b^2 + c^2 + 2bc \cosh A$	$\cosh a = \cosh b \cosh c + \sinh b \sinh c \cosh A$
$\cos b = \cos a \cos c + \sin a \sin c \cosh B$	$b^2 = a^2 + c^2 - 2ac \cosh B$	$\cosh b = \cosh a \cosh c - \sinh a \sinh c \cosh B$
$\cos c = \cos a \cos b + \sin a \sin b \cosh C$	$c^2 = a^2 + b^2 - 2ab \cosh C$	$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cosh C$
$\frac{\sin a}{\sinh A} = \frac{\sin b}{\sinh B} = \frac{\sin c}{\sinh C}$	$\frac{a}{\sinh A} = \frac{b}{\sinh B} = \frac{c}{\sinh C}$	$\frac{\sinh a}{\sinh A} = \frac{\sinh b}{\sinh B} = \frac{\sinh c}{\sinh C}$
$\cosh A = \cosh B \cosh C + \sinh B \sinh C \cos a$	$A = B + C$	$\cosh A = \cosh B \cosh C + \sinh B \sinh C \cosh a$
$\cosh B = \cosh A \cosh C - \sinh A \sinh C \cos b$	$B = A - C$	$\cosh B = \cosh A \cosh C - \sinh A \sinh C \cosh b$
$\cosh C = \cosh A \cosh B - \sinh A \sinh B \cos c$	$C = A - B$	$\cosh C = \cosh A \cosh B - \sinh A \sinh B \cosh c$

between the sphere, Euclidean and hyperbolic plane in the absolute trigonometry, the 'dual' constant κ_2 is restricted to a single particular (positive) value, so it cannot be given the attention it deserves when it is allowed to take on any real value. Furthermore, the usual definition of two-point homogeneity [24] excludes the degenerate Riemannian and pseudo-Riemannian spaces. In this respect, the explicit presence of κ_2 allows the consideration of the scheme within a complete *duality*, which otherwise would be hidden, showing up only in the spherical case. For instance, the dual of the hyperbolic geometry is the anti-de Sitter one, the natural metric of which is Lorentzian; simultaneous consideration of Riemannian and pseudo-Riemannian cases is therefore *essential* to fully display duality.

4.2. A compact notation

In this section we introduce a compact notation which allows us to get rid of the casual signs related to a, A. Let us denote the three sides as x_i, $i = 1, 2, 3$ and the three angles as X_I, $I = 1, 2, 3$ according to

$$x_1 = -a \qquad x_2 = b \qquad x_3 = c \qquad X_1 = -A \qquad X_2 = B \qquad X_3 = C. \tag{4.20}$$

With this notation, the basic equation (3.21) based in the vertex j can be written as

$$\mathrm{e}^{x_i P}\mathrm{e}^{X_K J}\mathrm{e}^{x_j P}\mathrm{e}^{X_I J}\mathrm{e}^{x_k P}\mathrm{e}^{X_J J} = 1 \tag{4.21}$$

for *any* cyclic permutation $i = I, j = J, k = K$ of the three indices 123. The triangular loop lateral excess δ (3.18) and angular excess Δ (3.16) appear in the present notation as the *symmetric* sums of the three 'oriented' sides or angles as

$$\delta = x_1 + x_2 + x_3 = -a + b + c \qquad \Delta = X_1 + X_2 + X_3 = -A + B + C. \tag{4.22}$$

It will also be convenient to replace the excesses by the quantities

$$e := \delta/2 \qquad E := \Delta/2 \tag{4.23}$$

and to introduce three other quantities as well as their three duals:

$$e_i := x_i - e \qquad E_I := X_I - E \tag{4.24}$$

which are related to the half sums p and P (4.11) by

$$\begin{array}{llll} e = p - a & e_1 = -p & e_2 = p - c & e_3 = p - b \\ E = P - A & E_1 = -P & E_2 = P - C & E_3 = P - B. \end{array} \tag{4.25}$$

Note that e_1 is different from zero and *negative*, while e_2, e_3 are (generically) different from zero and *positive*, just like the three sides x_1 and x_2, x_3. The same holds for the quantities related to angular excesses. If $\kappa_2 = 0$ then the three e_i reduce to the sides x_i ($\delta = e = 0$). Dually, if $\kappa_1 = 0$ then the three E_I reduce to the angles X_I ($\Delta = E = 0$).

In terms of this notation, the equations of trigonometry, including the alternative forms for the cosine theorems (4.12) and (4.13), can be written in a 'compact form' as

$$
\begin{array}{ll}
1\mathrm{i} & C_{\kappa_1}(x_i) = C_{\kappa_1}(x_j)C_{\kappa_1}(x_k) - \kappa_1 S_{\kappa_1}(x_j)S_{\kappa_1}(x_k)C_{\kappa_2}(X_I) \\
1\mathrm{I} & C_{\kappa_2}(X_I) = C_{\kappa_2}(X_J)C_{\kappa_2}(X_K) - \kappa_2 S_{\kappa_2}(X_J)S_{\kappa_2}(X_K)C_{\kappa_1}(x_i) \\
1''\mathrm{i} & 2S_{\kappa_1}(e)S_{\kappa_1}(e_i) = -\kappa_2 S_{\kappa_1}(x_j)S_{\kappa_1}(x_k)V_{\kappa_2}(X_I) \\
1''\mathrm{I} & 2S_{\kappa_2}(E)S_{\kappa_2}(E_I) = -\kappa_1 S_{\kappa_2}(X_J)S_{\kappa_2}(X_K)V_{\kappa_1}(x_i) \\
2 & \dfrac{S_{\kappa_1}(x_i)}{S_{\kappa_2}(X_I)} = \dfrac{S_{\kappa_1}(x_j)}{S_{\kappa_2}(X_J)} = \dfrac{S_{\kappa_1}(x_k)}{S_{\kappa_2}(X_K)} \\
3\mathrm{iJ} & S_{\kappa_1}(x_i)C_{\kappa_2}(X_J) = -C_{\kappa_1}(x_j)S_{\kappa_1}(x_k) - S_{\kappa_1}(x_j)C_{\kappa_1}(x_k)C_{\kappa_2}(X_I) \\
3\mathrm{Ij} & S_{\kappa_2}(X_I)C_{\kappa_1}(x_j) = -C_{\kappa_2}(X_J)S_{\kappa_2}(X_K) - S_{\kappa_2}(X_J)C_{\kappa_2}(X_K)C_{\kappa_1}(x_i) \\
4\mathrm{IJ} \equiv 4\mathrm{ij} & \kappa_2 S_{\kappa_2}(X_I)S_{\kappa_2}(X_J) - C_{\kappa_2}(X_I)C_{\kappa_2}(X_J)C_{\kappa_1}(x_k) \\
& \quad = \kappa_1 S_{\kappa_1}(x_i)S_{\kappa_1}(x_j) - C_{\kappa_1}(x_i)C_{\kappa_1}(x_j)C_{\kappa_2}(X_K).
\end{array}
\tag{4.26}
$$

When $\kappa_2 = 0$, the three equations $1''\mathrm{i}$ clearly imply $S_{\kappa_1}(e) = 0$, but the quotient $S_{\kappa_1}(e)/\kappa_2$ remains finite, and is given by

$$\frac{S_{\kappa_1}(e)}{\kappa_2} = -\frac{S_{\kappa_1}(x_j)S_{\kappa_1}(x_k)V_{\kappa_2}(X_I)}{2S_{\kappa_1}(e_i)}. \tag{4.27}$$

Dually, when $\kappa_1 = 0$, equations $1''\mathrm{I}$ lead to $S_{\kappa_2}(E) = 0$, but the quotient $S_{\kappa_2}(E)/\kappa_1$ remains finite:

$$\frac{S_{\kappa_2}(E)}{\kappa_1} = -\frac{S_{\kappa_2}(X_J)S_{\kappa_2}(X_K)V_{\kappa_1}(x_i)}{2S_{\kappa_2}(E_I)}. \tag{4.28}$$

4.3. Area and co-area and the dualities length/area and angle/co-area

The previous expressions show the natural appearance in this group-theoretical approach of the combinations $S_{\kappa_1}(e)/\kappa_2$ and $S_{\kappa_2}(E)/\kappa_1$ which are always well defined. By construction, the angular excess of a triangle loop is additive under decomposition of a triangle loop into two. Thus it is obvious that $S_{\kappa_2}(E)/\kappa_1$ is related to the triangular loop area. This relation is very well known in the two Riemannian spherical and hyperbolic geometries, where the standard expression for the absolute value $\mathcal{S}$ of the area enclosed by the triangle loop is easy to derive from the Gauss–Bonnet theorem and is related to the angular excess by $\kappa_1\mathcal{S} = \Delta$. This suggests a purely *group-theoretical* definition of *area* $\mathcal{S}$ and its dual quantity *co-area* s for triangle loops as

$$\mathcal{S} := \frac{\Delta}{\kappa_1} \qquad s := \frac{\delta}{\kappa_2}. \tag{4.29}$$

These definitions hold for all nine 2D CK spaces, no matter what the values of κ_1 or κ_2, and hence apply also to the pseudo-Riemannian and degenerate Riemannian CK spaces.

All appearances of $S_{\kappa_2}(E)/\kappa_1$ in the equations of trigonometry could be rewritten in terms of trigonometric functions of the area of the loop. The same happens dually for $S_{\kappa_1}(e)/\kappa_2$ in terms of co-area. In this rewriting, the label naturally associated with area is $\kappa_1^2\kappa_2$, while the co-area label is $\kappa_1\kappa_2^2$. This makes sense as area should be to the product P_1P_2 and co-area to

$J_{12}P_2$ what length—with label κ_1—is to P_1, and angle—with label κ_2—is to J_{12}. For the two basic sine and cosine functions of area and co-area we have

$$\begin{aligned} C_{\kappa_1^2\kappa_2}(\mathcal{S}) &:= C_{\kappa_2}(\Delta) = C_{\kappa_2}(2E) \qquad & S_{\kappa_1^2\kappa_2}(\mathcal{S}) &:= \frac{S_{\kappa_2}(\Delta)}{\kappa_1} = \frac{S_{\kappa_2}(2E)}{\kappa_1} \\ C_{\kappa_1\kappa_2^2}(s) &:= C_{\kappa_1}(\delta) = C_{\kappa_1}(2e) \qquad & S_{\kappa_1\kappa_2^2}(s) &:= \frac{S_{\kappa_1}(\delta)}{\kappa_2} = \frac{S_{\kappa_1}(2e)}{\kappa_2}. \end{aligned} \tag{4.30}$$

The length/area and angle/co-area dualities for the sphere have been recently discussed by Arnol'd [30] in a paper devoted to the geometry of spherical curves. These 'dualities' are indeed a *general* property for all nine CK geometries, and follow directly from the fundamental self-duality of the whole CK scheme (between lengths and angles), together with the 'transference' from angles E (or Δ) to areas $\mathcal{S}/2$ (or from lengths e (or δ) to co-areas $s/2$) implicitly contained in equations (4.30). However, while these dualities are present in all CK geometries, they are only clearly visible for the sphere, where by a suitable choice of length and angle units the constants κ_1 and κ_2 can be reduced to 1. In this spherical case the labels of either length, angle, area or co-area are all equal, so the transference from angle to area (or length to co-area) amounts to a simple equality between numerical values. In other CK geometries where some of the constants are negative, unveiling these dualities requires explicit use of a transference similar to those in (4.30).

By adding the area $\mathcal{S}$ and co-area s, to sides (x_i, e_i) and angles (X_I, E_I), all the basic equations can be written in a *minimal* form, with *no explicit* constants κ_1, κ_2, as

$$\begin{aligned} &1''\mathrm{i} \qquad & &S_{\kappa_1\kappa_2^2}(s/2)S_{\kappa_1}(e_i) = -S_{\kappa_1}(x_j)S_{\kappa_1}(x_k)S^2_{\kappa_2}(X_I/2) \\ &1''\mathrm{I} \qquad & &S_{\kappa_1^2\kappa_2}(\mathcal{S}/2)S_{\kappa_2}(E_I) = -S_{\kappa_2}(X_J)S_{\kappa_2}(X_K)S^2_{\kappa_1}(x_i/2) \\ &2 \qquad & &\frac{S_{\kappa_1}(x_i)}{S_{\kappa_2}(X_I)} = \frac{S_{\kappa_1}(x_j)}{S_{\kappa_2}(X_J)} = \frac{S_{\kappa_1}(x_k)}{S_{\kappa_2}(X_K)}. \end{aligned} \tag{4.31}$$

Two points are worth noting. First, the corresponding equations for any two particular spaces only differ by the *implicit* appearances of the constants κ_1, κ_2 (and also $\kappa_1^2\kappa_2, \kappa_1\kappa_2^2$) as labels of the trigonometric functions of sides, angles (and also area, co-area), respectively. This is reminiscent of the minimal coupling idea in general relativity: no explicitly dependent *curvature* terms should be introduced in the basic free equations when formulating the corresponding equation for a curved spacetime, but only those introduced implicitly through the connection (recall κ_1, κ_2 can indeed be interpreted as curvatures). Secondly, the only trigonometric function in these equations is the sine, which reduces to the variable itself when the label equals zero; the implementation of this kind of 'minimal coupling' to obtain the general equations (4.31) consists simply in replacing by their sines, each with the corresponding label, each term entering the 'purely flat' $\kappa_1 = 0, \kappa_2 = 0$ equations

$$\begin{aligned} &1''\mathrm{i} \qquad & &(s/2)e_i = -x_j x_k (X_I/2)^2 \\ &1''\mathrm{I} \qquad & &(\mathcal{S}/2)E_I = -X_J X_K (x_i/2)^2 \\ &2 \qquad & &\frac{x_i}{X_I} = \frac{x_j}{X_J} = \frac{x_k}{X_K}. \end{aligned} \tag{4.32}$$

4.4. A trigonometric bestiarium and some historical comments

Starting from the set of basic equations (4.31), we can easily derive a complete trigonometric *bestiarium* [10]; necessary trigonometric identities for this derivation can be found in the

Table 3. Equations of Euler, Gauss–Delambre–Mollweide and Napier.

Equations of Euler

$$S^2_{\kappa_1}\left(\tfrac{1}{2}x_i\right) = -\frac{S_{\kappa_1^2\kappa_2}(\mathcal{S}/2)S_{\kappa_2}(E_I)}{S_{\kappa_2}(X_J)S_{\kappa_2}(X_K)} \qquad S^2_{\kappa_2}\left(\tfrac{1}{2}X_I\right) = -\frac{S_{\kappa_1\kappa_2^2}(s/2)S_{\kappa_1}(e_i)}{S_{\kappa_1}(x_j)S_{\kappa_1}(x_k)}$$

$$C^2_{\kappa_1}\left(\tfrac{1}{2}x_i\right) = \frac{S_{\kappa_2}(E_J)S_{\kappa_2}(E_K)}{S_{\kappa_2}(X_J)S_{\kappa_2}(X_K)} \qquad C^2_{\kappa_2}\left(\tfrac{1}{2}X_I\right) = \frac{S_{\kappa_1}(e_j)S_{\kappa_1}(e_k)}{S_{\kappa_1}(x_j)S_{\kappa_1}(x_k)}$$

$$\frac{S_{\kappa_1}(x_i)}{S_{\kappa_2}(X_I)} = \frac{S_{\kappa_1}(x_j)}{S_{\kappa_2}(X_J)} = \frac{S_{\kappa_1}(x_k)}{S_{\kappa_2}(X_K)} = \frac{\{-S_{\kappa_1\kappa_2^2}(s/2)S_{\kappa_1}(e_i)S_{\kappa_1}(e_j)S_{\kappa_1}(e_k)\}^{1/2}}{\{-S_{\kappa_1^2\kappa_2}(\mathcal{S}/2)S_{\kappa_2}(E_I)S_{\kappa_2}(E_J)S_{\kappa_2}(E_K)\}^{1/2}}$$

Equations of Gauss–Delambre–Mollweide

$$\frac{S_{\kappa_2}\left(\frac{1}{2}(X_I+X_J)\right)}{S_{\kappa_2}\left(\frac{1}{2}X_K\right)} = -\frac{C_{\kappa_1}\left(\frac{1}{2}(x_i-x_j)\right)}{C_{\kappa_1}\left(\frac{1}{2}x_k\right)} \qquad \frac{C_{\kappa_2}\left(\frac{1}{2}(X_I+X_J)\right)}{C_{\kappa_2}\left(\frac{1}{2}X_K\right)} = \frac{C_{\kappa_1}\left(\frac{1}{2}(x_i+x_j)\right)}{C_{\kappa_1}\left(\frac{1}{2}x_k\right)}$$

$$\frac{S_{\kappa_2}\left(\frac{1}{2}(X_I-X_J)\right)}{S_{\kappa_2}\left(\frac{1}{2}X_K\right)} = \frac{S_{\kappa_1}\left(\frac{1}{2}(x_i-x_j)\right)}{S_{\kappa_1}\left(\frac{1}{2}x_k\right)} \qquad \frac{C_{\kappa_2}\left(\frac{1}{2}(X_I-X_J)\right)}{C_{\kappa_2}\left(\frac{1}{2}X_K\right)} = -\frac{S_{\kappa_1}\left(\frac{1}{2}(x_i+x_j)\right)}{S_{\kappa_1}\left(\frac{1}{2}x_k\right)}$$

Equations of Napier

$$\frac{T_{\kappa_2}\left(\frac{1}{2}(X_I+X_J)\right)}{T_{\kappa_2}\left(\frac{1}{2}X_K\right)} = -\frac{C_{\kappa_1}\left(\frac{1}{2}(x_i-x_j)\right)}{C_{\kappa_1}\left(\frac{1}{2}(x_i+x_j)\right)} \qquad \frac{T_{\kappa_2}\left(\frac{1}{2}(X_I-X_J)\right)}{T_{\kappa_2}\left(\frac{1}{2}X_K\right)} = -\frac{S_{\kappa_1}\left(\frac{1}{2}(x_i-x_j)\right)}{S_{\kappa_1}\left(\frac{1}{2}(x_i+x_j)\right)}$$

$$\frac{T_{\kappa_1}\left(\frac{1}{2}(x_i+x_j)\right)}{T_{\kappa_1}\left(\frac{1}{2}x_k\right)} = -\frac{C_{\kappa_2}\left(\frac{1}{2}(X_I-X_J)\right)}{C_{\kappa_2}\left(\frac{1}{2}(X_I+X_J)\right)} \qquad \frac{T_{\kappa_1}\left(\frac{1}{2}(x_i-x_j)\right)}{T_{\kappa_1}\left(\frac{1}{2}x_k\right)} = -\frac{S_{\kappa_2}\left(\frac{1}{2}(X_I-X_J)\right)}{S_{\kappa_2}\left(\frac{1}{2}(X_I+X_J)\right)}$$

Table 4. Some equations for the area.

Trigonometric functions of the area

$$C_{\kappa_1^2\kappa_2}(\mathcal{S}) = \frac{[1+C_{\kappa_1}(x_i)+C_{\kappa_1}(x_j)+C_{\kappa_1}(x_k)]^2 - 8C^2_{\kappa_1}(\frac{1}{2}x_i)C^2_{\kappa_1}(\frac{1}{2}x_j)C^2_{\kappa_1}(\frac{1}{2}x_k)}{8C^2_{\kappa_1}(\frac{1}{2}x_i)C^2_{\kappa_1}(\frac{1}{2}x_j)C^2_{\kappa_1}(\frac{1}{2}x_k)}$$

$$S_{\kappa_1^2\kappa_2}(\mathcal{S}) = -\frac{S_{\kappa_1}(x_i)S_{\kappa_1}(x_j)S_{\kappa_2}(X_K)[1+C_{\kappa_1}(x_i)+C_{\kappa_1}(x_j)+C_{\kappa_1}(x_k)]}{8C^2_{\kappa_1}(\frac{1}{2}x_i)C^2_{\kappa_1}(\frac{1}{2}x_j)C^2_{\kappa_1}(\frac{1}{2}x_k)}$$

Trigonometric functions of one-half the area

$$C_{\kappa_1^2\kappa_2}\left(\tfrac{1}{2}\mathcal{S}\right) = \frac{1+C_{\kappa_1}(x_i)+C_{\kappa_1}(x_j)+C_{\kappa_1}(x_k)}{4C_{\kappa_1}(\frac{1}{2}x_i)C_{\kappa_1}(\frac{1}{2}x_j)C_{\kappa_1}(\frac{1}{2}x_k)}$$

$$S_{\kappa_1^2\kappa_2}\left(\tfrac{1}{2}\mathcal{S}\right) = -\frac{S_{\kappa_2}(X_I)S_{\kappa_1}(x_j)S_{\kappa_1}(x_k)}{4C_{\kappa_1}(\frac{1}{2}x_i)C_{\kappa_1}(\frac{1}{2}x_j)C_{\kappa_1}(\frac{1}{2}x_k)} = \frac{\{-(S_{\kappa_1}(e)/\kappa_2)S_{\kappa_1}(e_i)S_{\kappa_1}(e_j)S_{\kappa_1}(e_k)\}^{1/2}}{2C_{\kappa_1}(\frac{1}{2}x_i)C_{\kappa_1}(\frac{1}{2}x_j)C_{\kappa_1}(\frac{1}{2}x_k)}$$

$$T_{\kappa_1^2\kappa_2}\left(\tfrac{1}{2}\mathcal{S}\right) = -\frac{S_{\kappa_2}(X_I)T_{\kappa_1}(\frac{1}{2}x_j)T_{\kappa_1}(\frac{1}{2}x_k)}{1-\kappa_1 C_{\kappa_2}(X_I)T_{\kappa_1}(\frac{1}{2}x_j)T_{\kappa_1}(\frac{1}{2}x_k)}$$

Trigonometric functions of one-fourth the area

$$C^2_{\kappa_1^2\kappa_2}\left(\tfrac{1}{4}\mathcal{S}\right) = \frac{C_{\kappa_1}(\frac{1}{2}e)C_{\kappa_1}(\frac{1}{2}e_i)C_{\kappa_1}(\frac{1}{2}e_j)C_{\kappa_1}(\frac{1}{2}e_k)}{C_{\kappa_1}(\frac{1}{2}x_i)C_{\kappa_1}(\frac{1}{2}x_j)C_{\kappa_1}(\frac{1}{2}x_k)}$$

$$S^2_{\kappa_1^2\kappa_2}\left(\tfrac{1}{4}\mathcal{S}\right) = -\frac{(S_{\kappa_1}(\frac{1}{2}e)/\kappa_2)S_{\kappa_1}(\frac{1}{2}e_i)S_{\kappa_1}(\frac{1}{2}e_j)S_{\kappa_1}(\frac{1}{2}e_k)}{C_{\kappa_1}(\frac{1}{2}x_i)C_{\kappa_1}(\frac{1}{2}x_j)C_{\kappa_1}(\frac{1}{2}x_k)}$$

$$T^2_{\kappa_1^2\kappa_2}\left(\tfrac{1}{4}\mathcal{S}\right) = -\frac{T_{\kappa_1}(\frac{1}{2}e)}{\kappa_2}T_{\kappa_1}\left(\tfrac{1}{2}e_i\right)T_{\kappa_1}\left(\tfrac{1}{2}e_j\right)T_{\kappa_1}\left(\tfrac{1}{2}e_k\right)$$

$$\frac{T_{\kappa_1}(\frac{1}{2}e_i)}{T_{\kappa_2}(\frac{1}{2}E_I)} = \frac{T_{\kappa_1}(\frac{1}{2}e_j)}{T_{\kappa_2}(\frac{1}{2}E_J)} = \frac{T_{\kappa_1}(\frac{1}{2}e_k)}{T_{\kappa_2}(\frac{1}{2}E_K)} = \frac{T_{\kappa_2}(\frac{1}{2}E)/\kappa_1}{T_{\kappa_1}(\frac{1}{2}e)/\kappa_2} = \frac{T_{\kappa_1^2\kappa_2}(\mathcal{S}/4)}{T_{\kappa_1\kappa_2^2}(s/4)}$$

appendix. We present the most relevant equations in tables 3 and 4 in a way which is simultaneously meaningful for *all* nine CK geometries.

We note that the sine theorem as reformulated in table 3 leads to the following inequalities that ensure a triangle in the CK space with constants κ_1, κ_2 exists:

$$\begin{aligned} &E > 0 \quad \text{when} \quad \kappa_1 > 0 \qquad && E < 0 \quad \text{when} \quad \kappa_1 < 0 \\ &e > 0 \quad \text{when} \quad \kappa_2 > 0 \qquad && e < 0 \quad \text{when} \quad \kappa_2 < 0. \end{aligned} \tag{4.33}$$

At this point, some historical comments seem to be pertinent. Trigonometry, motivated mainly in the spherical case by astronomy, has a very interesting and well documented history. A good authoritative reference for its historical development is the book by Rozenfel'd [31]; Ratcliffe [28] also contains historical notes. A standard reference for spherical trigonometry covering all the spherical versions of the equations we have presented here and much more (and which suitably reformulated also extends without exception to the nine geometries), is the book by Todhunter–Leathem [32]. Hyperbolic trigonometry was first satisfactorily settled by Lobachewski and Bolyai and is an essential tool when studying hyperbolic manifolds (see, for example, [27–29]).

The consideration of cosine and dual cosine theorems 1i and 1I as the basic equations of spherical trigonometric dates back to Euler, but the 'alternative' form 1′i is much older and essentially is the one given by Regiomontanus. The spherical cosine equations themselves are usually ascribed to Albategnius, and the first explicit appearance of the dual cosine equations is ascribed to Vieta, even though spherical polarity was clear to al-Ṭūsī in the 13th century. In table 3, the first set of formulae are the CK versions of the spherical formulae due to Euler, while the third set of equations are the general versions of Gauss–Delambre–Mollweide analogies (in the old meaning of proportion), and the last equations are the general version of the Napier analogies. In table 4, the formula for the sine of one-half area is known in the spherical case as Cagnoli's theorem, and the expression for the tangent of one-fourth the area of a spherical triangle in terms of the sides is due to L'Huillier, extending the Euclidean Heron–Archimedes area formula. Most formulae for spherical triangle area in terms of sides and/or angles were obtained by Euler. Other formulae for the trigonometric functions of one-half or one-fourth the area are also known in the spherical case, but bear no name; some are due to L'Huillier and Serret.

5. On the trigonometry of homogeneous spacetimes

In order to facilitate the reading of the general equations obtained here, in table 5 we present a sample of equations for the area in the six homogeneous spacetimes: the generalized formula of Cagnoli and that of Heron–L'Huillier as well as the last equation of table 4 that relates area and co-area. We use explicitly the *universe (time) radius* τ and the *relativistic constant* c; recall that the CK constants in the kinematic interpretation (2.8) of the CK space of points as $(1+1)$D spacetime are $\kappa_1 = \pm 1/\tau^2$ and $\kappa_2 = -1/c^2$. To stress that in the $(1+1)$D spacetime sides are *proper times* and angles are *rapidities*, we denote the (time) side lengths by τ_a, τ_b, τ_c, the angles by χ_A, χ_B, χ_C, and $\tau_p := (\tau_a + \tau_b + \tau_c)/2$, $\chi_P := (\chi_A + \chi_B + \chi_C)/2$. Should these equations be extended to include the $\kappa_2 > 0$ case (say, let $\kappa_2 = 1/c^2$), then c would play the role of a conversion constant between radians and the chosen angular measure.

A suitable view of the general trigonometric equations is as a kind of *deformation* of the *purely* linear equations (4.19) governed by the two constants κ_1, κ_2, which determine spacetime curvatures and/or signatures of the metric. From this viewpoint, it is clear that the good 'totally flat' reference 2D geometry is not the Euclidean one, but should be instead the Galilean one. The kinematic interpretation of the three basic trigonometric equations in Galilean

Table 5. Some equations involving the area $\mathcal{S}$ and co-area s of a timelike triangle in the six homogeneous spacetimes with $(\kappa_1=\pm 1/\tau^2, \kappa_2=-1/c^2)$.

Relative-time spacetimes	Absolute-time spacetimes
Anti-de Sitter $(+1/\tau^2, -1/c^2)$ $\sinh\left(\frac{\mathcal{S}}{2\tau^2 c}\right)=\frac{\sin(\tau_a/\tau)\sin(\tau_b/\tau)\sinh(\chi_C/c)}{4\cos(\tau_a/2\tau)\cos(\tau_b/2\tau)\cos(\tau_c/2\tau)}$ $\tanh^2\left(\frac{\mathcal{S}}{4\tau^2 c}\right)=-\tan\left(\frac{\tau_p}{2\tau}\right)\tan\left(\frac{\tau_p-\tau_a}{2\tau}\right)\tan\left(\frac{\tau_p-\tau_b}{2\tau}\right)\tan\left(\frac{\tau_p-\tau_c}{2\tau}\right)$ $\frac{\tanh(\mathcal{S}/4\tau^2 c)}{\tan(s/4\tau c^2)}=\frac{\tan(\tau_p/2\tau)}{\tanh(\chi_P/2c)}=\frac{\tan((\tau_p-\tau_b)/2\tau)}{\tanh((\chi_P-\chi_B)/2c)}=\frac{\tan((\tau_p-\tau_c)/2\tau)}{\tanh((\chi_P-\chi_C)/2c)}$	Oscillating NH $(+1/\tau^2, 0)$ $c=\infty$ $\mathcal{S}=\frac{\tau^2\sin(\tau_a/\tau)\sin(\tau_b/\tau)\chi_C}{2\cos(\tau_a/2\tau)\cos(\tau_b/2\tau)\cos(\tau_c/2\tau)}$ $\mathcal{S}^2=4s\tau^3\tan\left(\frac{\tau_p}{2\tau}\right)\tan\left(\frac{\tau_p-\tau_b}{2\tau}\right)\tan\left(\frac{\tau_p-\tau_c}{2\tau}\right)$ $\frac{\mathcal{S}}{s}=\frac{\tau\tan(\tau_p/2\tau)}{\frac{1}{2}\chi_P}=\frac{\tau\tan((\tau_p-\tau_b)/2\tau)}{\frac{1}{2}(\chi_P-\chi_B)}=\frac{\tau\tan((\tau_p-\tau_c)/2\tau)}{\frac{1}{2}(\chi_P-\chi_C)}$
Minkowskian $(0, -1/c^2)$ $\tau=\infty$ $\mathcal{S}=\frac{1}{2}\tau_a\tau_b\, c\sinh\left(\frac{\chi_C}{c}\right)$ $\mathcal{S}^2=-c^2\tau_p(\tau_p-\tau_a)(\tau_p-\tau_b)(\tau_p-\tau_c)$ $\frac{\mathcal{S}}{s}=\frac{\tau_p/2}{c\tanh(\chi_P/2c)}=\frac{(\tau_p-\tau_b)/2}{c\tanh((\chi_P-\chi_B)/2c)}=\frac{(\tau_p-\tau_c)/2}{c\tanh((\chi_P-\chi_C)/2c)}$	Galilean $(0, 0)$ $\tau=\infty, c=\infty$ $\mathcal{S}=\frac{1}{2}\tau_a\tau_b\chi_C$ $\mathcal{S}^2=\frac{1}{2}s\tau_p(\tau_p-\tau_b)(\tau_p-\tau_c)$ $\frac{\mathcal{S}}{s}=\frac{\tau_p}{\chi_P}=\frac{\tau_p-\tau_b}{\chi_P-\chi_B}=\frac{\tau_p-\tau_c}{\chi_P-\chi_C}$
de Sitter $(-1/\tau^2, -1/c^2)$ $\sinh\left(\frac{\mathcal{S}}{2\tau^2 c}\right)=\frac{\sinh(\tau_a/\tau)\sinh(\tau_b/\tau)\sinh(\chi_C/c)}{4\cosh(\tau_a/2\tau)\cosh(\tau_b/2\tau)\cosh(\tau_c/2\tau)}$ $\tanh^2\left(\frac{\mathcal{S}}{4\tau^2 c}\right)=-\tanh\left(\frac{\tau_p}{2\tau}\right)\tanh\left(\frac{\tau_p-\tau_a}{2\tau}\right)\tanh\left(\frac{\tau_p-\tau_b}{2\tau}\right)\tanh\left(\frac{\tau_p-\tau_c}{2\tau}\right)$ $\frac{\tanh(\mathcal{S}/4\tau^2 c)}{\tanh(s/4\tau c^2)}=\frac{\tanh(\tau_p/2\tau)}{\tanh(\chi_P/2c)}=\frac{\tanh((\tau_p-\tau_b)/2\tau)}{\tanh((\chi_P-\chi_B)/2c)}=\frac{\tanh((\tau_p-\tau_c)/2\tau)}{\tanh((\chi_P-\chi_C)/2c)}$	Expanding NH $(-1/\tau^2, 0)$ $c=\infty$ $\mathcal{S}=\frac{\tau^2\sinh(\tau_a/\tau)\sinh(\tau_b/\tau)\chi_C}{2\cosh(\tau_a/2\tau)\cosh(\tau_b/2\tau)\cosh(\tau_c/2\tau)}$ $\mathcal{S}^2=4s\tau^3\tanh\left(\frac{\tau_p}{2\tau}\right)\tanh\left(\frac{\tau_p-\tau_b}{2\tau}\right)\tanh\left(\frac{\tau_p-\tau_c}{2\tau}\right)$ $\frac{\mathcal{S}}{s}=\frac{\tau\tanh(\tau_p/2\tau)}{\chi_P/2}=\frac{\tau\tanh((\tau_p-\tau_b)/2\tau)}{(\chi_P-\chi_B)/2}=\frac{\tau\tanh((\tau_p-\tau_c)/2\tau)}{(\chi_P-\chi_C)/2}$

spacetime, where rapidities, defined as the canonical parameter of subgroups of pure inertial transformations, are equal to ordinary velocities, is clear:

$$\tau_a = \tau_b + \tau_c \qquad \chi_A = \chi_B + \chi_C \qquad \frac{\tau_a}{\chi_A} = \frac{\tau_b}{\chi_B} = \frac{\tau_c}{\chi_C}. \tag{5.1}$$

The first equation means that the (proper) time interval along any future timelike curve depends only on the endpoints, and corresponds to the *absolute time*; the same equation holds in both Newton–Hooke cases (see table 2). The second is the additivity, in Galilean spacetime, of the relative rapidities of three *non-concurrent* free motions; this also holds in the Minkowskian case, but not in the four curved spacetimes. This relation should not be confused with the additivity of relative rapidities for co-planar and concurrent free motions, which holds in *all* cases because rapidities are defined as canonical parameters of the one-parameter subgroup generated by $\mathcal{K}$. The third equation states that relative rapidities and time interval lengths in any triangle in Galilean spacetime are proportional; this is an absolutely elementary property of classical spacetime and holds only in this case. For the area and co-area of a Galilean triangle we have

$$\mathcal{S} = \tfrac{1}{2}\chi_A \tau_b \tau_c \qquad s = \tfrac{1}{2}\tau_a \chi_B \chi_C. \tag{5.2}$$

We recall that the co-area of a triangular loop in the six kinematic spaces is (proportional) to the difference of actions for a free particle following either of the two worldlines CB and CAB which determine the triangle loop [33].

These purely linear equations allow a *deformation* in two different senses [17], either by endowing spacetime with curvature $\kappa_1 \neq 0$ (obtaining the two Newton–Hooke spacetimes), or keeping it flat but introducing curvature, necessarily negative if causality must be preserved, in the space of timelike lines, described by the constant $\kappa_2 < 0$ (obtaining the Minkowskian spacetime of special relativity). If both processes are used simultaneously, we obtain the two de Sitter spacetimes.

This structural 'unfolding' of the complete CK scheme starting from its most degenerate case runs in a striking parallel with the historical development. Spacetime is nearly flat at the time and length of human scales: this *fact* lies behind classical physics. With hindsight, we can say that to assume a flat (i.e. $\kappa_1 = 0$) homogeneous model for the $(1 + 1)$D spacetime involved in 1D kinematics was natural. At the human (or even solar system) speed scale, the curvature in the space of uniform motions is also negligible, so to assume again a *flat* space of motions (or timelike lines), embodied in the equality $\kappa_2 = 0$ for the 1D kinematic group, was the only practical choice. Both assumptions greatly simplified (or rather, *allowed*) the linear mathematical description of classical physics. However, even at the homogeneous level of approximation, nature does not seem to be characterized by these non-generic choices. Relativity can be described as the discovery of a *negative* curvature in the space of 1D motions, and then the all-important relativistic constant c appears simply as related to the value of the curvature of this space of motions by $\kappa_2 = -1/c^2$. Special relativity still keeps a flat spacetime, another approximation which is abandoned, in a way much more general than by assuming it to be homogeneous, in the context of general relativity; if homogeneity is still kept, the possibilities are $\kappa_1 = \pm 1/\tau^2$.

Before spacetime geometry was under consideration, a similar situation arose for the physical 3D *space* geometry, whose characterization among the mathematical possibilities was at the root of Riemann's programme. General 3D CK spaces are parametrized by *three* CK constants, say μ_1, μ_2, μ_3 and correspond to a space with constant curvature μ_1 and whose metric is reducible at each point to $\mathrm{diag}(1, \mu_2, \mu_2\mu_3)$; restriction to a locally Euclidean space is embodied in the choices $\mu_2 > 0$, $\mu_3 > 0$. In the $(1 + 3)$D homogeneous spacetime

$SO_{\kappa_1,\kappa_2,\kappa_3,\kappa_4}(5)/SO_{\kappa_2,\kappa_3,\kappa_4}(4)$ [14, 15], whose labels are $\kappa_1 = \pm 1/\tau^2$, $\kappa_2 = -1/c^2$, $\kappa_3 = 1$, $\kappa_4 = 1$, the 3-space orthogonal to the fiducial timelike line through the origin (and hence all 3-spaces orthogonal to any timelike line) can be identified with the homogenous space $SO_{\kappa_1\kappa_2,\kappa_3,\kappa_4}(4)/SO_{\kappa_3,\kappa_4}(3)$. Hence the constants μ_1, μ_2, μ_3 are given by $\mu_1 = \kappa_1\kappa_2$, $\mu_2 = \kappa_3$, $\mu_3 = \kappa_4$ so that the curvature of the 3-space orthogonal to a given timelike direction is the product $\kappa_1\kappa_2 = \mp 1/(\tau^2 c^2)$. This means that in the non-relativistic spacetimes, even if spacetime is curved, the 3-space is *flat*. In the three relativistic cases, the 3-space is only flat in the Minkowskian case, but is curved in the anti-de Sitter (space curvature $-1/(\tau^2 c^2)$) and de Sitter spacetimes (space curvature $1/(\tau^2 c^2)$). In both cases the universe radius R, with dimensions of space-length, is given by $R = c\tau$.

Here we also have a clear example of *hidden universal constants*, in the sense given to this term by Lévy-Leblond [34]: both κ_3 and κ_4 are not usually considered as universal constants only because the fact that they are *non-zero and positive* allows us to make them apparently disappear by reducing them to the value 1 (thus making plane angles and dihedral space angles apparently dimensionless). Once performed, this reduction forbids further consideration of these constants and the possibility that they *could* be either zero or negative in other conceivable but still homogeneous spaces is simply out of sight. The character of $\kappa_1\kappa_2$ as a possible universal constant was understood much earlier: Lobachewski explored the possibility of physical 3-space geometry being hyperbolic (i.e. negatively curved), and tried to give experimental bounds to a constant he called k (in modern terms, the curvature would be $-1/k^2$) for our physical space under the assumption that light travels along geodesics in this physical 3-space; the argument is based on the existence of a *minimum* parallax (for a given baseline) even for infinitely distant stars [7, 35]. The founding fathers of hyperbolic geometry could have hardly imagined that the geometry they were discovering/inventing was indeed realized by nature and to a good approximation, not as the geometry of space itself, but as the geometry of the space of uniform motions.

6. Concluding remarks

Although we have not covered any applications here, we should point to the relevance of many of the complicated trigonometric equations whose 'general' form we have derived in several fields. For instance, they appear in the Zamolodchikov solution for tetrahedral equations as factorization conditions for the S-matrix in (1 + 2)D [36, 37], both reproduced in [38]. The extension of the Moyal-type formulation of quantum mechanics to spaces with constant curvature also involves many of the complicated equations for spherical or hyperbolic area in terms of sides. A recent paper by Jing-Ling Chen and Mo-Lin Ge [39] identifies the Wigner angle of the rotation appearing in the product of pure Lorentz transformations to the defect of a triangle in hyperbolic geometry; this and analogous results can also be deduced from our approach. Analogous results involving triangle defects appear in relation to geometrical phases. Triangles in anti-de Sitter and de Sitter spaces also display properties similar to the 'parallelism angle' found in hyperbolic geometry, and are related to the existence of horizons.

As far as we know, the approach we have given to the trigonometry of the real CK spaces is new, and we have also obtained some results apparently unknown on the trigonometry of several spacetimes. In spite of this, we feel that the main value of this paper is to display in this simplest case (the symmetric rank-one homogeneous spaces of real type and their limiting spaces) the potentialities this group-theoretical approach to trigonometry has for the study of many other interesting spaces whose trigonometry is still unknown. Pursuing this line, the trigonometry of complex, quaternionic and octonionic type CK spaces will be discussed in a forthcoming companion paper [23]; only the complex case is really relevant, as the others reduce directly to the complex case. Although primarily mathematical, the study of trigonometry in Hermitian complex spaces has a very direct and deep connection with physics, the link being the geometry of the quantum space of states [40, 41].

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Appendix. Some relations for the trigonometric functions

The main identities for the trigonometric functions defined in (2.11)–(2.13) depending on a (curvature) label κ and involving one or two arbitrary arguments x, y are given by [10]

$$C_\kappa^2(x) + \kappa S_\kappa^2(x) = 1 \tag{A.1}$$

$$C_\kappa(x) = 1 - \kappa V_\kappa(x) \tag{A.2}$$

$$C_\kappa(2x) = C_\kappa^2(x) - \kappa S_\kappa^2(x) \tag{A.3}$$

$$S_\kappa(2x) = 2S_\kappa(x)C_\kappa(x) \tag{A.4}$$

$$C_\kappa^2\left(\tfrac{1}{2}x\right) = \tfrac{1}{2}(C_\kappa(x) + 1) \tag{A.5}$$

$$S_\kappa^2\left(\tfrac{1}{2}x\right) = \frac{1 - C_\kappa(x)}{2\kappa} = \tfrac{1}{2}V_\kappa(x) \tag{A.6}$$

$$T_\kappa\left(\tfrac{1}{2}x\right) = \frac{1 - C_\kappa(x)}{\kappa S_\kappa(x)} = \frac{S_\kappa(x)}{C_\kappa(x) + 1} \tag{A.7}$$

$$C_\kappa(x \pm y) = C_\kappa(x)C_\kappa(y) \mp \kappa S_\kappa(y)S_\kappa(x) \tag{A.8}$$

$$S_\kappa(x \pm y) = S_\kappa(x)C_\kappa(y) \pm S_\kappa(y)C_\kappa(x) \tag{A.9}$$

$$V_\kappa(x \pm y) = V_\kappa(x) + V_\kappa(y) - \kappa V_\kappa(x)V_\kappa(y) \pm S_\kappa(x)S_\kappa(y) \tag{A.10}$$

$$T_\kappa(x \pm y) = \frac{T_\kappa(x) \pm T_\kappa(y)}{1 \mp \kappa T_\kappa(x)T_\kappa(y)} \tag{A.11}$$

$$C_\kappa(x) + C_\kappa(y) = 2C_\kappa\left(\tfrac{1}{2}(x + y)\right)C_\kappa\left(\tfrac{1}{2}(x - y)\right) \tag{A.12}$$

$$C_\kappa(x) - C_\kappa(y) = -2\kappa S_\kappa\left(\tfrac{1}{2}(x + y)\right)S_\kappa\left(\tfrac{1}{2}(x - y)\right) \tag{A.13}$$

$$S_\kappa(x) \pm S_\kappa(y) = 2S_\kappa\left(\tfrac{1}{2}(x \pm y)\right)C_\kappa\left(\tfrac{1}{2}(x \mp y)\right). \tag{A.14}$$

Let x, y, z be three arbitrary real numbers and the quantities defined by

$$\begin{aligned} p &= \tfrac{1}{2}(x + y + z) \qquad & p - x &= \tfrac{1}{2}(y + z - x) \\ p - y &= \tfrac{1}{2}(x + z - y) \qquad & p - z &= \tfrac{1}{2}(x + y - z). \end{aligned} \tag{A.15}$$

Then we find the following identities involving three arbitrary arguments:

$$C_\kappa(x + y) + C_\kappa(z) = 2C_\kappa(p)C_\kappa(p - z) \tag{A.16}$$

$$C_\kappa(x - y) + C_\kappa(z) = 2C_\kappa(p - x)C_\kappa(p - y) \tag{A.17}$$

$$C_\kappa(x + y) - C_\kappa(z) = -2\kappa S_\kappa(p)S_\kappa(p - z) \tag{A.18}$$

$$C_\kappa(x - y) - C_\kappa(z) = 2\kappa S_\kappa(p - x)S_\kappa(p - y) \tag{A.19}$$

$$S_\kappa(x+y)+S_\kappa(z)=2S_\kappa(p)C_\kappa(p-z) \tag{A.20}$$

$$S_\kappa(x-y)+S_\kappa(z)=2S_\kappa(p-y)C_\kappa(p-x) \tag{A.21}$$

$$S_\kappa(x+y)-S_\kappa(z)=2S_\kappa(p-z)C_\kappa(p) \tag{A.22}$$

$$S_\kappa(x-y)-S_\kappa(z)=-2S_\kappa(p-x)C_\kappa(p-y) \tag{A.23}$$

$$V_\kappa(x+y)-V_\kappa(z)=2S_\kappa(p)S_\kappa(p-z) \tag{A.24}$$

$$V_\kappa(x-y)-V_\kappa(z)=-2S_\kappa(p-x)S_\kappa(p-y) \tag{A.25}$$

$$C_\kappa(x)S_\kappa(y)=C_\kappa(p)S_\kappa(p-z)+S_\kappa(p-x)C_\kappa(p-y) \tag{A.26}$$

$$S_\kappa(x)S_\kappa(y)=S_\kappa(p)S_\kappa(p-z)+S_\kappa(p-x)S_\kappa(p-y) \tag{A.27}$$

$$\begin{aligned}4C_\kappa\left(\tfrac{1}{2}x\right)C_\kappa\left(\tfrac{1}{2}y\right)C_\kappa\left(\tfrac{1}{2}z\right)&-[1+C_\kappa(x)+C_\kappa(y)+C_\kappa(z)]\\ &=8\kappa^2S_\kappa\left(\tfrac{1}{2}p\right)S_\kappa\left(\tfrac{1}{2}(p-x)\right)S_\kappa\left(\tfrac{1}{2}(p-y)\right)S_\kappa\left(\tfrac{1}{2}(p-z)\right)\end{aligned} \tag{A.28}$$

$$\begin{aligned}4C_\kappa\left(\tfrac{1}{2}x\right)C_\kappa\left(\tfrac{1}{2}y\right)C_\kappa\left(\tfrac{1}{2}z\right)&+[1+C_\kappa(x)+C_\kappa(y)+C_\kappa(z)]\\ &=8C_\kappa\left(\tfrac{1}{2}p\right)C_\kappa\left(\tfrac{1}{2}(p-x)\right)C_\kappa\left(\tfrac{1}{2}(p-y)\right)C_\kappa\left(\tfrac{1}{2}(p-z)\right)\end{aligned} \tag{A.29}$$

$$\begin{aligned}4\kappa^2S_\kappa(p)S_\kappa(p-x)&S_\kappa(p-y)S_\kappa(p-z)\\ &=16C_\kappa^2\left(\tfrac{1}{2}x\right)C_\kappa^2\left(\tfrac{1}{2}y\right)C_\kappa^2\left(\tfrac{1}{2}z\right)-\left[1+C_\kappa(x)+C_\kappa(y)+C_\kappa(z)\right]^2\\ &=1-C_\kappa^2(x)-C_\kappa^2(y)-C_\kappa^2(z)+2C_\kappa(x)C_\kappa(y)C_\kappa(z)\end{aligned} \tag{A.30}$$

$$\begin{aligned}4C_\kappa(p)C_\kappa(p-x)&C_\kappa(p-y)C_\kappa(p-z)\\ &=-1+C_\kappa^2(x)+C_\kappa^2(y)+C_\kappa^2(z)+2C_\kappa(x)C_\kappa(y)C_\kappa(z).\end{aligned} \tag{A.31}$$

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